Tutorial from 28.05.2020

I am willing to answer your questions and comments.

You can gain points for indicating non-trivial mistakes in the notes!

Homework deadline: **24:00 on Wednesday 03.06**

1 Solutions of the homework problems

Exercise 1. *Show that the Ellentuck topology is Baire: if $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a countable family of dense open subsets of $[\mathbb{N}]^{\aleph_0}$ (in the Ellentuck topology) then the intersection $\bigcap_{n\in\mathbb{N}}\mathcal{U}_n$ is also dense (in the Ellentuck topology).*

Consider a family $(\mathcal{U}_n)_{n\in\mathbb{N}}$ as in the statement. Take $s < A$, we will construct $X \in [s, A]$ such that $X \in \bigcap_{n\in\mathbb{N}}\mathcal{U}_n$. Without loss of generality assume that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$.

Put $s_0 = s$ and $A_0 = A$. For $i = 0, \ldots$ assuming that $s_i < A_i$ is defined, apply density of $\mathcal{U}_i$ to learn that there exists a basic open set $[s'_i, A'_i] \subseteq [s_i, A_i]$ such that $[s'_i, A'_i] \subseteq \mathcal{U}_i$. Let $n_i = \min(A'_i)$ and let $s_{i+1} = s'_i \cup \{n_i\}$, while $A_{i+1} = A_i - \{n_i\}$. Then also $[s_{i+1}, A_{i+1}] \subseteq \mathcal{U}_i$. Notice that $s_0 \subsetneq s_1 \ldots$ Take $X = \bigcup_{i\in\mathbb{N}} s_i$. By the construction we know that for every $i \in \mathbb{N}$ we have $X \in [s_i, A_i]$ and therefore $X \in \bigcap_{n\in\mathbb{N}}\mathcal{U}_n$. As $s_0 = s$ and $A_0 = A$ we also know that $X \in [s, A]$.

Exercise 2 ($\star$)**.** *Let X be a Polish space and $f \colon [\mathbb{N}]^{\aleph_0} \to X$ be a Borel function (in the normal topology on $[\mathbb{N}]^{\aleph_0}$). Show that there exists $H \in [\mathbb{N}]^{\aleph_0}$ such that $f\restriction_{[H]^{\aleph_0}}$ is continuous in the normal topology of $[\mathbb{N}]^{\aleph_0}$.*

This is based on a solution by Damian Głodkowski.

Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of X. We will inductively construct sets $A_0 \supseteq A_1 \supseteq \ldots$ and a sequence $a_0 < a_1 < \ldots$ such that $a_n \in A_n$ for $n \in \mathbb{N}$. Then, putting $H = \{a_n \mid n \in \mathbb{N}\}$ will give the desired set H.

We will say that $[s, B]$ is *homogeneous* for U_n if $[s, B] \subseteq f^{-1}(U_n)$ or $[s, B] \cap f^{-1}(U_n) = \varnothing$. Since $f^{-1}(U_0)$ is Borel, it is completely Ramsey, so there exists a set B such that $[\varnothing, B]$ is homogeneous for U_0. Let $A_0 = B$ and a_0 be any element of A_0.

Assume that $A_0, \ldots, A_n$ and $a_0, \ldots, a_n$ are defined.

Let $\mathsf{P}\big(\{a_0, \dots, a_n\}\big) = \{s_i \mid i < 2^{n+1}\}$. As $f^{-1}(U_{n+1})$ is Borel so it is completely Ramsey. Thus, one can inductively construct a sequence of infinite sets $A_n/a_n \supseteq B_0 \supseteq \dots \supseteq B_{2^{n+1}-1}$ such that for each i we have $[s_i, B_i]$ is homogeneous for U_{n+1}. Let $A_{n+1} = B_{2^{n+1}-1}$ and let a_{n+1} be any element of A_{n+1} (clearly $a_{n+1} > a_n$). In particular, for each i we have $[s_i, A_{n+1}] \subseteq [s_i, B_i]$ and therefore $[s_i, A_{n+1}]$ is homogeneous for U_{n+1}.

Take $H = \{a_n \mid n \in \mathbb{N}\}$. Thus, it is enough to show that for each n the set $f^{-1}(U_n) \cap [H]^{\aleph_0}$ is open in $[H]^{\aleph_0}$. Notice that

$$[H]^{\aleph_0} = \bigcup_{s \in \mathsf{P}\left(\{a_0, \dots, a_{n-1}\}\right)} [s, H/a_{n-1}].$$

However, for each $s \in \mathsf{P}\big(\{a_0, \dots, a_{n-1}\}\big)$ we have $[s, H/a_{n-1}] \subseteq [s, A_n]$ which means that $[s, H/a_{n-1}]$ is homogeneous for U_n. This implies that $f^{-1}(U_n) \cap [H]^{\aleph_0}$ is a finite union of sets of the form $[s, H/a_{n-1}]$. However, these sets are open in $[H]^{\aleph_0}$ in the standard topology. Thus $f^{-1}(U_n) \cap [H]^{\aleph_0}$ is open in $[H]^{\aleph_0}$ as a finite union of open sets.

2 New material

2.1 Completely Ramsey sets vs. Baire Property

This section replies more accurately to the questions that were raised for the last Q&A session. Thanks to prof. Zakrzewski for clarifying some of these topics.

First, the following two theorems from the lecture notes show that in the Ellentuck topology, being Ramsey and Baire Property go hand-in-hand.

Theorem 2.1 (Page 19 of WZTM 11). *A set is Ramsey null iff it is meagre in the Ellentuck topology.*

Theorem 2.2 (Ellentuck, Page 8 of WZTM 11). *A set is completely Ramsey iff it has Baire Property in the Ellentuck topology.*

BP vs. Ramsey In general, having Baire Property in the usual topology and in the Ellentuck topology are two different things.

One direction can be witnessed by the following example. Take any set that is not Ramsey $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$. Let E be the set of all even natural numbers.

Consider

$$\mathcal{U}' = \left\{X \in [E]^{\aleph_0} \mid \{n \mid 2n \in X\} \in \mathcal{U}\right\}.$$

Then $\mathcal{U}' \subseteq [E]^{\aleph_0}$ and the latter set is meagre in $[\mathbb{N}]^{\aleph_0}$ (and of measure 0 in 2^ω). However, $\mathcal{U}'$ is not completely Ramsey, because one cannot find a homogeneous subset in $[\varnothing, E]$.

Another direction: a completely Ramsey set without BP can be constructed based on Exercise 17.

Set-theoretical assumptions Under the assumption that $V = L$ (a set theoretical assumption that is equiconsistent with ZFC) there exists a Bernstein set in $\mathbf{\Delta}^1_2([\mathbb{N}]^{\aleph_0})$. It is easy to see that such a set is not Ramsey (otherwise it would either contain or be disjoint with a copy of the Cantor set).

On the other hand, assuming Projective Determinacy (that guarantees e.g. that all projective sets have Baire Property) one also gets that they are all completely Ramsey (Harrington, Kechris [1981], see also Exercise 38.19 in Kechris).

2.2 Material from WZTM-12

In the current lecture notes standard Borel spaces are being introduced.

As a particular example, the Effros Borel space is given. For that, recall the notions of $K(X)$ and $F(X)$, see Section 1 from tutorials from 12.03.2020.

At the end of page 4 a representation of a certain set as $\bigcap_{n\in\mathbb{N}} U_n$ for $\overline{U_{n+1}} \subseteq U_n$ open is given.

Exercise 3. *Take a Polish space X. When a set $A \subseteq X$ can be represented as $\bigcap_{n\in\mathbb{N}} U_n$ with $\overline{U_{n+1}} \subseteq U_n$ and all U_n open?*

To prove that Effros Borel space is always standard, one invokes a map:

$$E \overset{c}{\mapsto} \big(n \mapsto [E \cap U_n \neq \varnothing]\big) \in 2^\omega,$$

where the square bracket equals 1 if the given condition is true.

Exercise 4. *Show that for every set $E \subseteq X$ we have $c(E) = c(\overline{E})$.*

Exercise 5. *Prove that $c\restriction_{F(X)}$ is injective.*

Then one takes $Y = c(F(X))$ and notices that $c\restriction_{F(X)}$ is an isomorphism between the measurable spaces $\langle F(X), \mathcal{E}\rangle$ and $\langle Y, \mathcal{B}(Y)\rangle$. Our aim is to show that Y itself is Borel in 2^ω.

Exercise 6. *Check that the set G defined in mid page 6 is in fact a G_δ.*

Exercise 7. *Check that $Y \subseteq G$.*

Thus, it remains to see that $G \subseteq Y$. For that one takes $x \in G$ and defines a set $E \subseteq X$ as containing certain points of X (defined relatively to x), see upper half of page 7.

Exercise 8. *Verify that $x = c(E)$.*

Another approach to the above problem can be based on the following representation of Polish spaces.

Exercise 9. *Let X be a Polish space. Show that X is homeomorphic to a* **closed** *subset of $\mathbb{R}^\omega$.*

Hint: we already know that every Polish space is homeomorphic to a G_δ subset of $[0,1]^\omega$. Thus, take $X = \bigcap_{n\in\mathbb{N}} U_n$ in $[0,1]^\omega$ and do something smart :)

For that approach to work, one needs additionally to know the following.

Exercise 10. *Let X be a Polish space. Show that $K(X)$ is Borel in $F(X)$.*

Hint: express the condition of being totally bounded using basic sets $[U]_{F(X)}$. Instead of saying that something covers K, joggle with basic open sets as in the definition of G from Exercise 6.

2.3 Uniformisation results

The next step is a famous result of Ryll-Nardzewski. The basic idea is simple: choose the left-most branch of every tree. However, to adjust the general situation to the case of trees, based on the following lemma.

Exercise 11. *How the inductive subsets $U_{s^\frown i}$ are defined for a given U_s in Lemma 1 on page 9?*

Exercise 12. *Check that the function f defined as in the proof of Lemma 1 on page 9 is continuous, open, and surjective.*

Then comes the proof of Kuratowski–Ryll–Nardzewski, based on the function c that picks the left-most branch of the tree that corresponds to the given closed subset of $\mathbb{N}^\omega$.

Exercise 13. *Prove that $c(F) \in F$ for the function c defined as at the top of page 11 of lecture notes.*

Then one checks that the function c defined that way is Borel by checking the following.

Exercise 14. *Verify equation just above Case 2 on page 11 of lecture notes.*

Then one composes c with the (lifting of) appropriate surjective map from $\mathbb{N}^\omega$ onto our given Polish space. This gives rise to a function d, see the bottom of page 11.

Exercise 15. *Check that $d(F) \in F$ for each non-empty closed set $F \in F(X)$.*

3 New homework

This time please choose at most one problem to solve!

Exercise 16. *Carefully solve Exercise 12: check that the function f defined in Lemma 1 on page 9 of lecture notes WZTM-12 is continuous, open, and surjective. Please write down all the details, no "hand-waving" arguments :)*

Exercise 17 ($\star$)**.** *Construct a set $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ that is Ramsey null but comeagre in the standard topology on $[\mathbb{N}]^{\aleph_0}$.*

4 Old homework

In parallel you can still solve the following problem from one of the previous homeworks. Solving it does not count to the restrictions of "at most one problem".

Exercise 18 ($\star$)**.** *Assume that X is a non-empty perfect Polish space and $R \subseteq X^2$ is a comeagre set. Prove that there exists a Cantor set $C \subseteq X$ and a comeagre set $D \subseteq X$ such that $C \times D \subseteq R$.*

As a hint, you may focus on constructing $C \subseteq X$ more than on finding $D \subseteq Y$: begin by finding a Cantor set C such that the set

$$\{y \in Y \mid \forall x \in C.\ (x, y) \in R\}$$

is dense in Y.

Another hint is as follows: if $C \subseteq X$ is a Cantor set then it is compact. We know something about projections along compact coordinates, no?

5 Hints

Hint to Exercise 3 The answer is closed. First, let A be closed. Take $U_n = \bigcup_{x \in A} B(x, 2^{-n})$. Clearly U_n is open and $\bigcap_{n \in \mathbb{N}} U_n = A$. It remains to see that $\overline{U_{n+1}} \subseteq U_n$. Take any point $y \in \overline{U_{n+1}}$. It means that there exists $y' \in U_{n+1}$ such that $d(y, y') < 2^{-n-1}$. But then there exists $x \in A$ such that $d(y', x) < 2^{-n-1}$. Thus, $d(y, x) < 2^{-n}$, which implies that $y \in U_n$.

Now assume that $A = \bigcap_{n \in \mathbb{N}} U_n$ as above. However, by the assumptions on U_n, we know that $A = \bigcap_{n \in \mathbb{N}} \overline{U_n}$ and as an intersection of closed sets A must itself be closed.

Hint to Exercise 4 Take any coordinate $n \in \mathbb{N}$ and notice that $E \cap U_n$ is non-empty if and only if $\overline{E} \cap U_n$ is non-empty (recall that $\overline{E}$ is the intersection of all closed sets containing E).

Hint to Exercise 5 Take two distinct closed sets $F \neq F'$. Without loss of generality assume that there exists $x \in F - F'$. But then there exists a basic open ball U_n such that $x \in U_n$ and $U_n \cap F' = \varnothing$. Therefore, $c(F)(n) = 1$, while $c(F')(n) = 0$.

Hint to Exercise 6 First notice that the conditions like $U_n \subseteq U_m$ or $\overline{U_m} \subseteq U_n$ do not involve x and have no influence on its topological complexity — from the point of view of x these are just facts about numbers. Thus, the first part of the definition of G is equivalent to

$$\bigcap_{(n,m) \in S} \{x \in 2^\omega \mid x(n) = 0\} \cup \{x \in 2^\omega \mid x(m) = 1\},$$

where $S = \{(n, m) \mid U_n \subseteq U_m\}$ is fixed. This piece is closed (notice that we changed the implication into a union).

The second piece can be written in the same manner (you can put the condition $x(n) = 1$ inside the existential quantifier). Then the formula gets the shape

$$\bigcap_{(n,k)\in S'} \bigcup_{m\in S''_{n,k}} \{\ldots\},$$

for properly defined S' and $S''_{n,k}$. The inner sets above are again clopen so the whole set is a G_δ.

Hint to Exercise 7 Take any point $y \in Y$ as witnessed by $F \in F(X)$ such that $c(F) = y$. We need to show that $y \in G$. We can do it separately, for the two conditions of G. Take $n, m \in \mathbb{N}$ such that $U_n \subseteq U_m$. Assume that $y(n) = 1$, what means that $F \cap U_n \neq \varnothing$. But then $F \cap U_m \neq \varnothing$ and therefore $y(m) = 1$ as well.

Now consider the second condition and let $n, k \in \mathbb{N}$ with $k > 0$ be such that $y(n) = 1$. Thus, $F \cap U_n \neq \varnothing$. Let $x \in F \cap U_n$ witness that. Similarly as in Exercise 3 there exists a basic open set U_m such that $x \in U_m \subseteq \overline{U_m} \subseteq U_n$ and $\operatorname{diam}(U_m) \leqslant \frac{1}{k}$ by taking sufficiently small ball around x. Then $F \cap U_m \neq \varnothing$ because of x and the inner condition of G holds.

Hint to Exercise 8 Recall that $x \in G$ is any point. Take any coordinate $n \in \mathbb{N}$, we need to check that $x(n) = c(E)(n)$. First assume that $x(n) = 1$. By inductively applying the second part of the definition of G we obtain a sequence of coordinates ℓ_i with $\ell_0 = n$ satisfying the first three conditions from the definition of E on page 7. Thus, the intersection $\bigcap_i U_{\ell_i} = \{z\}$ is a member of E. This means that $c(E)(n) = 1$, because $\ell_0 = n$ so $z \in U_n \cap E$.

Now assume that $c(E)(n) = 1$, as witnessed by $z \in E \cap U_n$. Let ℓ_i be a sequence of numbers witnessing the fact that $z \in E$. In particular, for each i we have $x(\ell_i) = 1$. By taking sufficiently large i (the second item implies that the indices ℓ_i must be arbitrarily big) we know that $\operatorname{diam}(U_{\ell_i}) < \frac{1}{\ell_i}$ and so $U_{\ell_i} \subseteq U_n$. Thus, the first part of the definition of G guarantees that as $x(\ell_i) = 1$ also $x(n) = 1$.

Hint to Exercise 9 See Kechris 4.17.

Put $F_n = [0,1]^\omega - U_n$ closed. Define a function $f\colon X \to \mathbb{R}^\omega$ by $f(x)(2n) = x_n$, i.e. on even coordinates it is just an embedding. For an odd coordinate let $f(x)(2n{+}1) = \frac{1}{d(x,F_n)}$. Clearly f is injective. It is also easy to check that

f is open. Thus, it remains to see that the image of X under f is closed in $\mathbb{R}^\omega$.

Assume that $(x_k)_{k\in\mathbb{N}}$ is a sequence of points in X such that $\lim_{k\to\infty} f(x_k) = y$ exists. The even coordinates of $f(x_k)$ provide a unique candidate x for the limit of x_k. It remains to see that $x \in X$ to know that $y = f(x) \in f(X)$. However, since the sequence $\big(f(x_k)\big)_{k\in\mathbb{N}}$ is convergent, the odd coordinates of $f(x_k)$ must be bounded, so for each n the distances $d(x_k, F_n)$ are separated from 0. Therefore, $x \in U_n$ and $x \in X$.

Hint to Exercise 10 Fix a countable basis U_n of X that consists of balls of rational diameters around some countable dense subset of X.

Consider the following definition:

$$K(X) = \{F \in F(X) \mid \forall k \in \mathbb{N}.\ \exists s \in [\mathbb{N}]^{<\aleph_0}.\ \forall n \in s.\ \mathrm{diam}(U_n) < 2^{-k} \wedge \mathrm{cov}(s, F)\},$$

where $\mathrm{cov}(s, F)$ is defined as:

$$\forall i \in \mathbb{N}.\ F \cap U_i \neq \varnothing \Rightarrow \exists n \in s.\ U_i \cap U_n \neq \varnothing.$$

If $\bigcup_{n\in s} U_n \supseteq F$ then clearly $\mathrm{cov}(s, F)$ holds. Thus, if K is compact then it satisfies the above definition.

Assume that a set $F \in F(X)$ satisfies the above definition. We want to prove that it is totally bounded. Take $\varepsilon > 0$ and let k be such that $2^{-k} < \varepsilon$. Let s be a family of indices given by the above definition for k. Fix $s' \in [\mathbb{N}]^{<\aleph_0}$ that is obtained from s by replacing in s each index n of a basic ball U_n by a bigger ball (but still with diameter smaller than ε). We claim that $\bigcup_{n\in s'} U_n \supseteq F$, which concludes the proof.

Assume contrarily that there exists $x \in F - \bigcup_{n\in s'} U_n$. But then there exists a ball U_i around x that is disjoint from all the sets U_n for $n \in s$ — because we've enlarged their radii to obtain s'. This violates the conditions above, because $U_i \cap U_n = \varnothing$ for each $n \in s$.

Now, the condition $\mathrm{cov}(s, F)$ (for a fixed set $s \in [\mathbb{N}]^{<\aleph_0}$) is a countable intersection of complements of sets of the form $[U_i]_{F(X)}$. Thus, the above definition of $K(X)$ is something like $\mathbf{\Pi}^0_3$: intersection of unions of intersections of complements of basic open sets.

Hint to Exercise 11 One can cover the set U_s by the countable family of balls of radius at most $2^{-\mathrm{lh}(s)-1}$, each contained in U_s with its own closure.

Hint to Exercise 13 Take $F \in F(X)$ non-empty. Let $x = c(F)$ as in the definition of c. Notice that for each $n \in \mathbb{N}$ the fact that $x_n = k$ is witnessed by some point $x_n \in F \cap N_{x|n\hat{\ }k}$. Moreover, $\lim_{n\to\infty} x_n = x$. Since F is closed and all $x_n \in F$, we know that also $x \in F$.

Hint to Exercise 14 We have $s \in \mathbb{N}^{<\omega}$ and $n = \mathrm{lh}(s)$.

First take $F \in c^{-1}(N_s)$. Let $x = c(F)$ with $s \preceq x$. We know that $x \in F$ and therefore $F \cap N_s \neq \varnothing$. Consider $t \in \mathbb{N}^n$ and assume for the sake of contradiction that $t <_{\mathrm{lex}} s$ and $F \cap N_t \neq \varnothing$. Then one easily gets a contradiction with the definition of x, because for some $n' < n$ we must have violated the minimality requirement.

Now assume that F satisfies the given conditions. Then one inductively shows that in the definition of $x = c(F)$ the first n steps must equal $s_0, s_1, \ldots, s_{n-1}$. Therefore, $x \in N_s$ witnesses that $F \in c^{-1}(N_s)$.

Hint to Exercise 15 Take non-empty $F \in F(X)$. Then one has $T = f^{-1}(F) \in F(\mathbb{N}^\omega)$ given by $\Phi(F)$. By applying c one obtains $c(T) \in T$. Then $f(c(T))$ must belong to F, by the choice of T. But $f(c(T)) = d(F)$.