Tutorial from 21.05.2020

I am willing to answer your questions and comments.

You can gain points for indicating non-trivial mistakes in the notes!

Homework deadline: **24:00 on Wednesday 27.05**

1 Solutions of the homework problems

This time please choose at most one problem to solve!

Exercise 1. *Show that if $\mathcal{U}_0$ and $\mathcal{U}_1$ are completely Ramsey then $\mathcal{U}_0 \cup \mathcal{U}_1$ is also completely Ramsey.*

We will check it directly from the definition. Consider $s < B$. First use the fact that $\mathcal{U}_0$ is completely Ramsey to find $B_0 \in [B]^{\aleph_0}$ such that $[s, B_0] \subseteq \mathcal{U}_0$ or $[s, B_0] \cap \mathcal{U}_0 = \varnothing$. In the first case $[s, B_0] \subseteq \mathcal{U}_0 \cup \mathcal{U}_1$ and we are done, so assume that $[s, B_0] \cap \mathcal{U}_0 = \varnothing$. Use the fact that $\mathcal{U}_1$ is completely Ramsey for $s < B_0$ to find $B_1 \in [B_0]^{\aleph_0}$ such that either $[s, B_1] \subseteq \mathcal{U}_1$ or $[s, B_1] \cap \mathcal{U}_1 = \varnothing$. Again in the first case we are done. If $[s, B_1] \cap \mathcal{U}_1 = \varnothing$ (while also $[s, B_1] \cap \mathcal{U}_0 = \varnothing$) then $[s, B_1] \cap (\mathcal{U}_0 \cup \mathcal{U}_1) = \varnothing$ and we are done as well.

Exercise 2 ($\star$)**.** *Prove that there exists a family of ascending subsets $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \ldots$ of $[\mathbb{N}]^{\aleph_0}$ such that each of the sets $\mathcal{U}_i \subseteq [\mathbb{N}]^{\aleph_0}$ is Ramsey but their union $\mathcal{U} \overset{def}{=} \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ is not Ramsey.*

Let $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ be any set that is not Ramsey. For $n \in \mathbb{N}$ let

$$\mathcal{U}_n = \{A \in \mathcal{U} \mid \min A \leqslant n\}.$$

Clearly $\mathcal{U}_n$ is an ascending family of sets. Moreover, as each element of $\mathcal{U}$ has certain minimum, $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n = \mathcal{U}$ is not Ramsey. It remains to prove that for every $n \in \mathbb{N}$ the set $\mathcal{U}_n$ is Ramsey. However, $\big[\{n+1, n+2, \ldots\}\big]^{\aleph_0} \cap \mathcal{U}_n = \varnothing$.

2 New material

We continue the previous exercises studying the lecture notes on Ellentuck topology. Again, our aim is the fact that every Borel set in $[\mathbb{N}]^{\aleph_0}$ (in the usual topology) is Ramsey.

Recall the notions of $s < A$, etc from the previous exercises. In particular, for $s < A$ we have

$$[s, A] \stackrel{\text{def}}{=} \{X \in [\mathbb{N}]^{\aleph_0} \mid s \subseteq X \subseteq (s \cup A)\}.$$

See page 5 of WZTM-11.

Based on that, one defines (see page 7 of WZTM-11) the Ellentuck topology on $[\mathbb{N}]^{\aleph_0}$ with basic open sets all the sets of the form $[s, A]$ for $s < A$.

Exercise 3. *Show that the Ellentuck topology extends the standard topology.*

Exercise 4. *Show that for any $s < A$ and $r < B$ either $[s, A] \cap [r, B] = \varnothing$ or $[s, A] \cap [r, B] = [s \cup r, A \cap B]$.*

Since $[\varnothing, \mathbb{N}] = [\mathbb{N}]^{\aleph_0}$, it shows that the above family in fact generates certain topology extending the standard one.

The following fact shows that locally the Ellentuck topology has countable neighbourhoods.

Exercise 5. *Show that for every point $X \in [\mathbb{N}]^{\aleph_0}$ there exists a countable family of basic open sets $\mathcal{U}_n \subseteq [\mathbb{N}]^{\aleph_0}$ (in the Ellentuck topology) such that $\bigcap_{n\in\mathbb{N}} \mathcal{U}_n = \{X\}$.*

The following fact implies that the Ellentuck topology is not separable (no countable dense subset).

Exercise 6. *Show that in the Ellentuck topology there exists a family of cardinality continuum of non-empty open subsets of $[\mathbb{N}]^{\aleph_0}$ that are pairwise disjoint.*

Entail that this topology is not Polish.

Exercise 7. *Show that the Ellentuck topology is Baire: if $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a countable family of dense open subsets of $[\mathbb{N}]^{\aleph_0}$ (in the Ellentuck topology) then the intersection $\bigcap_{n\in\mathbb{N}} \mathcal{U}_n$ is also dense (in the Ellentuck topology).*

Hint: you can check from the definition: take $s < A$ and construct a set $X \in [s, A]$ such that $\forall n \in \mathbb{N}.\ X \in \mathcal{U}_n$.

Then the important result of Ellentuck is invoked: $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ is completely Ramsey iff it has Baire Property in the Ellentuck topology.

From that one entails that analytic sets (in the standard topology) are completely Ramsey, because of WZTM-7 p. 11 (see the beginning of page 9 in WZTM-11).

Then come the observations that we have discussed previously (pages 9–13).

Now an important Lemma 1 on page 13 says that for any $s < A$ the whole space $[\mathbb{N}]^{\aleph_0}$ is homeomorphic to $[s, A]$. Thus, one can transfer topological properties of $[\mathbb{N}]^{\aleph_0}$ to any local neighbourhood. It is then used in the proof of Lemma 2 on page 14 to lift the fact that $\mathcal{U}$ is Ramsey to the fact that $\mathcal{U}$ is completely Ramsey.

Exercise 8. *Show that the family of nowehere dense subsets of* $[\mathbb{N}]^{\aleph_0}$ *(in the Ellentuck topology) is a* σ*-ideal.*

This all leads to the notion of a *Ramsey null* set $\mathcal{U}$ (page 18): for every $s < A$ there exists $B \in [A]^{\aleph_0}$ with $[s, B] \cap \mathcal{U} = \varnothing$. Clearly if $\mathcal{U}$ is Ramsey null then it is nowhere dense (in the Ellentuck topology). But the converse also holds (Theorem on page 19): if $\mathcal{U}$ is Ramsey null then it is meagre in the Ellentuck topology. This fact is used in the final proof of Ellentuck's theorem (see pages 19–20).

2.1 Set that is not Ramsey

The following simple example was proposed by Dominika Regiec.

Take $X \subseteq 2^\omega$ as the set from Exercise 1 from 23.04.2020 — X contains exactly one from each pair of $\alpha, \alpha' \in 2^\omega$ that differ on exactly one (or equivalently an odd number) of positions. This set is constructed by choosing a single member from each equivalence class of the relation $\sim$ defined as $\alpha \sim \alpha'$ if they differ on a finite number of positions; then we extend it to the rest of the equivalence classes by even-differences.

Notice that $[\mathbb{N}]^{\aleph_0}$ as a subset of 2^ω coincides with $2^\omega - [0^\omega]_\sim$: the equivalence class of the sequence of only-zeros is exactly $[\mathbb{N}]^{<\aleph_0}$.

Let $\mathcal{U} = [\mathbb{N}]^{\aleph_0} \cap X$ for the set X as above. Assume for the sake of contradiction that $\mathcal{U}$ is Ramsey, i.e. $[A]^{\aleph_0} \subseteq \mathcal{U}$ (the dual case is analogous, considering the complement of $\mathcal{U}$). But since $A \in \mathcal{U}$ then $A - \{\min A\} \notin \mathcal{U}$, because they differ on a single position. However, $A - \{\min A\} \in [A]^{\aleph_0}$, a contradiction.

2.2 Rosenthal's lemma

An important application of Galvin-Prikry theorem is explained in Kechris 19.E. Let S be any set and $\ell^\infty(S)$ be the Banach space of bounded functions $f\colon S \to \mathbb{R}$ with the norm $|f|_\infty = \sup_{s\in S} |f(s)|$.

Theorem 2.1 (Rosenthal)**.** *If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\ell^\infty(S)$ then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that is either pointwise convergent (for every $s \in S$ the limit $\lim_{k\to\infty} f_{n_k}(s)$ exists); or there are positive constants $a, b > 0$ such that for any $n \in \mathbb{N}$ and any $c_0, \ldots, c_{n-1} \in \mathbb{R}$ we have*

$$a \sum_i |c_i| \leqslant \left| \sum_i c_i \cdot f_{n_i} \right|_\infty \leqslant b \sum_i |c_i|.$$

The second possibility above says that the functions f_{n_i} (up to some fixed constants $a, b > 0$) span a subspace of $\ell^\infty(S)$ in the same way as unit vectors span the spaces $\mathbb{R}^n$ with the norm $|x|_1 \stackrel{\text{def}}{=} \sum_i |x_i|$ for $x \in \mathbb{R}^n$ $(n = 1, 2, \ldots)$.

A rather direct application of this theorem is expressed by Corollary 19.21 of Kechris — please parse it :) It is a dychotomy theorem, that either a real Banach space X embeds ℓ^1 or a variant of Banach-Alaoglu theorem holds: the ball is weakly compact.

To prove Rosenthal's theorem, one reduces it to the following lemma about sequences of pairs of sets. First, we say that a pair (A, B) of subsets of S is *disjoint* if $A \cap B = \varnothing$.

1. A sequence of disjoint pairs is *independent* if for every two finite disjoint subsets $F, G \subseteq \mathbb{N}$ we have

$$\bigcap_{n\in F} A_n \cap \bigcap_{n\in G} B_n \neq \varnothing.$$

It means that although each single pair (A_n, B_n) is disjoint, when n varies then the sets overlap (one can take for instance $F = \{n\}$ and $G = \{m\}$ for $n \neq m$ and obtain that $A_n \cap B_m \neq \varnothing$).

2. A sequence of disjoint pairs is *convergent* if for every $x \in X$ either: (for almost all n we have $x \notin A_n$) or (for almost all n we have $x \notin B_n$).

Lemma 2.2 (Lemma 19.24 in Kechris). *Every sequence of disjoint pairs either contains a convergent subsequence or an independent subsequence.*

This is obtained via an application of Galvin-Prikry theorem. Define $P \subseteq [\mathbb{N}]^{\aleph_0}$ by

$$\{n_0 < n_1 < \ldots\} \in P \Leftrightarrow \forall k. \left[\bigcap_{i<k, i \text{ even}} A_{n_i} \cap \bigcap_{i<k, i \text{ odd}} B_{n_i} \neq \varnothing \right].$$

Exercise 9. *Show that P is closed in $[\mathbb{N}]^{\aleph_0}$ with the usual topology.*

By Galvin-Prikry we obtain an infinite subset $H \subseteq \mathbb{N}$ such that either $[H]^{\aleph_0} \subseteq P$ or $[H]^{\aleph_0} \cap P = \varnothing$. Let $H = \{m_0 < m_1 < \ldots\}$.

In the first case (see the proof in Kechris) the subsequence $\big((A_{m_{2i+1}}, B_{m_{2i+1}})\big)$ is independent (notice that all the indices here are odd).

In the second case the subsequence $\big((A_{m_i}, B_{m_i})\big)$ is convergent.

3 New homework

This time please choose at most one problem to solve!

Exercise 10. *Solve Exercise 7.*

Exercise 11 ($\star$). *Let X be a Polish space and $f \colon [\mathbb{N}]^{\aleph_0} \to X$ be a Borel function (in the normal topology on $[\mathbb{N}]^{\aleph_0}$). Show that there exists $H \in [\mathbb{N}]^{\aleph_0}$ such that $f\restriction_{[H]^{\aleph_0}}$ is continuous in the normal topology of $[\mathbb{N}]^{\aleph_0}$.*

4 Old homework

In parallel you can still solve the following problem from one of the previous homeworks. Solving it does not count to the restrictions of "at most one problem".

Exercise 12 ($\star$). *Assume that X is a non-empty perfect Polish space and $R \subseteq X^2$ is a comeagre set. Prove that there exists a Cantor set $C \subseteq X$ and a comeagre set $D \subseteq X$ such that $C \times D \subseteq R$.*

As a hint, you may focus on constructing $C \subseteq X$ more than on finding $D \subseteq Y$: begin by finding a Cantor set C such that the set

$$\{y \in Y \mid \forall x \in C.\ (x, y) \in R\}$$

is dense in Y.

Another hint is as follows: if $C \subseteq X$ is a Cantor set then it is compact. We know something about projections along compact coordinates, no?

5 Hints

Hint to Exercise 3 We have already seen it in Exercise 3 of the previous tutorials.

Hint to Exercise 4 First, if $A \cap B$ is finite then clearly $[s, A] \cap [r, B] = \varnothing$. So assume that $A \cap B \in [\mathbb{N}]^{\aleph_0}$. Now if neither $s \subseteq r$ nor $r \subseteq s$ then let $n \in s \backslash r$ and $n' \in r \backslash s$ and by the symmetry let $n < n'$. If $X \in [s, A] \cap [r, B]$ then $n \in X$ but it is a contradiction, as $n \notin r$ while $n < \max(r)$ so $n \notin B$ which means that $X \notin [r, B]$. Thus in that case we also have $[s, A] \cap [r, B] = \varnothing$.

Therefore, assume by the symmetry that $s \subseteq r$. Now it is easy to check that $[s, A] \cap [r, B] = [r, A \cap B]$ and $r = s \cup r$.

Hint to Exercise 5 It is enough to take $\mathcal{U} = [s_n, A_n]$ for $s_n = X \cap \{0, \ldots, n\}$ and $A_n = \mathbb{N} - \{0, \ldots, n\}$.

Hint to Exercise 6 Fix a bijection ι between $\mathbb{N}$ and $2^{<\omega}$. For each infinite branch $\pi \in 2^{\omega}$ take $A_\pi = \iota^{-1}(\{s \in 2^{<\omega} \mid s \prec \pi\})$. Notice that if $\pi \neq \pi'$ then $A_\pi \cap A_{\pi'}$ is finite. Thus, $[\varnothing, A_\pi] \cap [\varnothing, A_{\pi'}] = \varnothing$. As there is continuum possible branches π, the family $\big([\varnothing, A_\pi]\big)_{\pi \in 2^{\omega}}$ has the desired properties.

Hint to Exercise 8 First observe that if $F \subseteq F'$ and F' is nowhere dense (i.e. $\overline{(}F')$ has empty interior) then also F is nowhere dense. Thus it is enough to show that this family is closed under countable unions. But this follows from Lemma 3 on page 16 that says that any countable union of nowhere dense sets is nowhere dense itself.

Hint to Exercise 9 Notice that if $\{n_0 < n_1 < \ldots\} \notin P$ then it is witnessed by some $k \in \mathbb{N}$ and for some $s \in 2^k$ we have $N_s \cap [\mathbb{N}]^{\aleph_0} \cap P = \varnothing$.