Tutorial from 14.05.2020

I am willing to answer your questions and comments.

You can gain points for indicating non-trivial mistakes in the notes!

Homework deadline: **24:00 on Wednesday 20.05**

1 Solutions of the homework problems

Exercise 1. *Prove that if $B \subseteq X \times Y$ is Borel and all its sections B_x for $x \in X$ are at most countable then the projection $\pi_X(B)$ is Borel itself.*

First apply the theorem of Luzin–Novikov (WZTM-9 p. 13) to observe that B is a countable union of Borel sets F_n in $X \times Y$ such that for each $x \in X$ we have $|(F_n)_x| \leqslant 1$. Clearly $\pi_X(B) = \bigcup_{n\in\mathbb{N}} \pi_X(F_n)$, so it is enough to show that each of the sets $\pi_X(F_n)$ is Borel in X.

Take $n \in \mathbb{N}$. Apply the theorem of Luzin–Souslin (WZTM-5 p. 10) to the function $f = \pi_X : X \times Y \to X$ and the set $F_n \subseteq X \times Y$. This function is continuous, so in particular Borel. Moreover, it is injective on the Borel set F_n. Therefore, $\pi_X(F_n)$ is Borel in X.

There is a totally different and quite fancy solution by Mateusz Przyborowski, but it invokes a theorem not covered in this lecture (see Kechris, Theorem 18.11).

Exercise 2 ($\star$)**.** *Show that there exists a partition of $[2^\omega]^2$ into infinitely many pieces $(P_i)_{i\in\mathbb{N}}$ such that the pieces are clopen (i.e. P_i^* is clopen in $\{(x,y) \in 2^\omega \times 2^\omega \mid x \neq y\}$) and this partition does not admit any homogeneous copy of the Cantor set.*

Let P_n for $n \in \mathbb{N}$ contain a pair $\{\alpha, \beta\}$ for $\alpha \neq \beta$ if $\Delta(\alpha,\beta) \stackrel{\text{def}}{=} \min\{n \mid \alpha(n) \neq \beta(n)\} = n$. Clearly the sets P_n form a partition of $[2^\omega]^2$. Moreover, P_n^* is clopen in $\{(\alpha,\beta) \in 2^\omega \times 2^\omega \mid \alpha \neq \beta\}$ as the sum of the basic sets $N_{0^n{}^\frown 0} \times N_{0^n{}^\frown 1} \cup N_{0^n{}^\frown 1} \times N_{0^n{}^\frown 0}$.

Now assume for the sake of contradiction that there exists an n-homogeneous set A of cardinality greater than 2: $A \supseteq \{\alpha_0, \alpha_1, \alpha_2\}$. But then by the pigeonhole principle either $\alpha_0(n) = \alpha_1(n)$, or $\alpha_1(n) = \alpha_2(n)$, or $\alpha_0(n) = \alpha_2(n)$, contradicting the fact that A is homogeneous for P_n.

2 New material

This time the lecture notes provide a technical development aiming at the following theorem:

Theorem 2.1. *Every Borel set in* $[\mathbb{N}]^{\aleph_0}$ *is Ramsey.*

This is achieved via a bit of a detour, with invocation of the so-called Ellentuck topology. The following two tutorials will help you digest that material in a bottom-up fashion: we begin with the down-to=/the-ground combinatorics and then wrap it up into the more abstract topological argument.

For the following notation see page 5 of lecture notes.

First, $[A]^{\aleph_0}$ is the family of all infinite subsets of A, while $[A]^{<\aleph_0}$ contains all finite subsets of A. Elements of $[\mathbb{N}]^{\aleph_0}$ will be denoted using capital letters A, X, Y, Z, while the elements of $[\mathbb{N}]^{<\aleph_0}$ using lowercase letters s, r, t. Thus, when $B \in [X]^{\aleph_0}$ then it means that $B \subseteq X$ and B is infinite.

We treat $[\mathbb{N}]^{\aleph_0}$ as a G_δ subset of 2^ω, identified with the sequences containing infinitely many 1s (this set used to be denote G at some early stage of our journey). Thus, $[\mathbb{N}]^{\aleph_0}$ is a Polish space with the topology generated by the sets of the form $N_s \cap [\mathbb{N}]^{\aleph_0}$. This topology is explained at the beginning of subsection 11.2 of lecture notes, see pages 2–3.

Exercise 3. *Notice that the above basic sets of* $[\mathbb{N}]^{\aleph_0}$ *are of the form*

$$\{A \in [\mathbb{N}]^{\aleph_0} \mid \forall i \leqslant n.\ i \in A \Leftrightarrow i \in r\},$$

for some $n \in \mathbb{N}$ *and* $r \subseteq \{0, \dots, n\} \in [\mathbb{N}]^{<\aleph_0}$.

In other words, the basic sets of the topology on $[\mathbb{N}]^{\aleph_0}$ require some small numbers to **belong** or **not belong** to A, leaving all the rest unspecified.

From the topological perspective, we treat elements $A \in [\mathbb{N}]^{\aleph_0}$ as points (however, it makes sense to say that $A \subseteq B$ for a pair of points $A, B \in [\mathbb{N}]^{\aleph_0}$). We think of sets $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ as subsets of our topological space, thus it makes sense to say that $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ is Borel or analytic.

For the sake of clarity, we put $\max(\varnothing) = -1$ when treating $\varnothing \in [\mathbb{N}]^{<\aleph_0}$. For $n \in \mathbb{N} \cup \{-1\}$, $s \in [\mathbb{N}]^{<\aleph_0}$, and $A \in [\mathbb{N}]^{\aleph_0}$ we write:

- $s < A$ if $\max(s) < \min(A)$,
- $A/n = \{m \in A \mid m > n\}$,

- if $s < A$ then $[s, A] \stackrel{\text{def}}{=} \{X \in [\mathbb{N}]^{\aleph_0} \mid s \subseteq X \subseteq (s \cup A)\}$.

As mentioned in lecture notes (again see page 5), $X \in [s, A]$ iff $X \cap \{0, \ldots, \max(s)\} = s$ and all other elements of X come from A — notice relationship with Exercise 3.

2.1 Ramsey sets

Now we move to mid of page 3 of lecture notes to study the concept of Ramsey sets.

Definition 2.2. *A set $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ is* Ramsey *if there exists a set $A \in [\mathbb{N}]^{\aleph_0}$ such that either $[A]^{\aleph_0} \subseteq \mathcal{U}$ or $[A]^{\aleph_0} \cap \mathcal{U} = \varnothing$.*

The very nice example on page 1 of the lecture notes defines

$$\mathcal{U} = P_0 = \left\{A \in [\mathbb{N}]^{\aleph_0} \mid A = \min_{\preceq}\left([A]^{\aleph_0}\right)\right\}$$

for some well-order $\preceq$ of $[\mathbb{N}]^{\aleph_0}$ (such a well-order could be used to define a well-ordering of $\mathbb{R}$ and therefore requires some variant of the axiom of choice). Then it is proved (first half of page 2) that this set is not Ramsey. Please digest it!

Exercise 4. *The family of all sets $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ that are Ramsey is closed under complement: if $\mathcal{U}$ is Ramsey then $[\mathbb{N}]^{\aleph_0} - \mathcal{U}$ is also Ramsey.*

As it turns out, sets that are Ramsey are not closed under union: there are $\mathcal{U}_0$ and $\mathcal{U}_1$ Ramsey with $\mathcal{U}_0 \cup \mathcal{U}_1$ not Ramsey, for that example on page 4 is used. The example goes as follows: we identify $\mathbb{N}$ with $\{0, 1\} \times \mathbb{N}$ and fix a subset $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ that is **not** Ramsey. Based on that, we define $\mathcal{U}_0 = \{\{0\} \times A \mid A \in \mathcal{U}\} \subseteq [\{0, 1\} \times \mathbb{N}]^{\aleph_0}$ and $\mathcal{U}_1 = \{\{1\} \times A \mid A \notin \mathcal{U}\} \subseteq [\{0, 1\} \times \mathbb{N}]^{\aleph_0}$.

Exercise 5. *Show that both $\mathcal{U}_0$ and $\mathcal{U}_1$ are Ramsey in $[\{0, 1\} \times \mathbb{N}]^{\aleph_0}$.*

Exercise 6. *Show that $\mathcal{U}_0 \cup \mathcal{U}_1$ is* **not** *Ramsey in $[\{0, 1\} \times \mathbb{N}]^{\aleph_0}$.*

2.2 Completely Ramsey sets

As often in mathematics, to prove more we need to require more. Here "more" means *completely Ramsey* sets. Such set is in a sense *everywhere Ramsey*: $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ is *completely Ramsey* if for every $s < B$ there exists $A \in [B]^{\aleph_0}$ such that $[s, A] \subseteq \mathcal{U}$ or $[s, A] \cap \mathcal{U} = \varnothing$.

[see page 6 of lecture notes for the definition of completely Ramsey sets]

Exercise 7. *Show that if $\mathcal{U}$ is completely Ramsey then it is Ramsey.*

Notice that $r \in 2^{<\omega}$ encodes certain finite set $\bar{r} \subseteq \mathbb{N}$ of positions i where $r(i) = 1$. However, contrary to the case of $[s, A]$, such $r \in 2^{<\omega}$ requires that some positions do not belong to elements of N_r. In particular, for $r = <0, 1, 1, 0, 1, 0>$, we have $\bar{r} = \{1, 2, 4\}$ and $\max(\bar{r}) = 4$ but r additionally requires that 5 does not belong to any member of N_r. Notice that in that example $|r| = 6$ and always $\bar{r} \subseteq \{0, \dots, |r| - 1\}$.

The above fact is claimed at the end of page 6 of lecture notes.

Exercise 8. *Take $r \in 2^{<\omega}$ and let $\mathcal{U} = N_r \cap [\mathbb{N}]^{\aleph_0}$. Show that $\mathcal{U}$ is completely Ramsey.*

In general it follows from the lecture notes that the family of sets that are completely Ramsey forms a σ-algebra. We will see it later. For now, we can prove something weaker.

Exercise 9. *Show that if $\mathcal{U}_0$ and $\mathcal{U}_1$ are completely Ramsey then $\mathcal{U}_0 \cup \mathcal{U}_1$ is also completely Ramsey.*

Exercise 10. *Show that if $\mathcal{U}_0$ and $\mathcal{U}_1$ are completely Ramsey then $\mathcal{U}_0 \cap \mathcal{U}_1$ is also completely Ramsey.*

The next portion of lecture notes (on pages 7, 8) regards Ellentuck topology and is left for the next tutorials.

2.3 Accept or reject

We are back to combinatorics in the middle of page 9 of lecture notes. We consider fixed (and known from the context) set $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$. Take $s < Y$ and say that:

- Y *accepts* s if $[s, Y] \subseteq \mathcal{U}$,
- Y *rejects* s if for every $X \in [Y]^{\aleph_0}$ we have $[s, X] \nsubseteq \mathcal{U}$.

Exercise 11 (Observation 1 from page 9)**.** *If Y accepts (rejects) s and $Z \in [Y]^{\aleph_0}$ then Z accepts (rejects) s.*

In other words, think about a fixed s, and smaller and smaller subsets $X \in [Y]^{\aleph_0}$. Then:

1. Y accepts s if all these subsets X have the property that $[s, X] \subseteq \mathcal{U}$ (which boils down to saying that $[s, Y] \subseteq \mathcal{U}$, as Y is the biggest of possible $X \in [Y]^{\aleph_0}$);

2. Y can be undecided yet, when $[s, Y] \nsubseteq \mathcal{U}$ but for some $X \in [Y]^{\aleph_0}$ we have $[s, X] \subseteq \mathcal{U}$ — not all hope is lost, and we might shrink Y to X to make it accept s;

3. Y rejects s if all hope is lost, because no matter which $X \in [Y]^{\aleph_0}$ we choose, there exists $A \in [s, X]$ such that $A \notin \mathcal{U}$ (the choice of A depends on the choice of X).

Exercise 12 (Observation 2 from page 9)**.** *Given any $s < Y$ there exists $X \in [Y]^{\aleph_0}$ such that either X accepts s or X rejects s.*

Exercise 13 (Observation 3 from page 10)**.** *Given any Y and $s_1, \ldots, s_n$ with $s_i < Y$ for $i = 1, \ldots, n$ there exists $X \in [Y]^{\aleph_0}$ such that for each $i = 1, \ldots, n$ either X accepts s_i or X rejects s_i (it might accept for some i and reject for other).*

Then there comes Observation 4 from page 10 which constructs a set Z (via a kind of diagonal argument) such that for every $s \in [Z]^{<\aleph_0}$ either $Z/\max(s)$ rejects s of $Z/\max(s)$ accepts s. Such Z is called *decisive*.

Please study that construction carefully, as it will reappear!

This all leads to Observation 5 from page 11 which shows how to make a decisive set even better. This proof again uses *diagonalisation* as in Observation 4 (I'm not entirely sure about the word "diagonalisation", but it does have such a flavour...).

The rest of the lecture notes is devoted to certain facts about the Ellentuck topology, which are based on Observation 5 and related arguments. We leave it for the next week :)

3 New homework

This time please choose at most one problem to solve!

Exercise 14. *Solve Exercise 9.*

Exercise 15 ($\star$)**.** *Prove that there exists a family of ascending subsets $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \ldots$ of $[\mathbb{N}]^{\aleph_0}$ such that each of the sets $\mathcal{U}_i \subseteq [\mathbb{N}]^{\aleph_0}$ is Ramsey but their union $\mathcal{U} \stackrel{def}{=} \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ is not Ramsey.*

4 Old homework

In parallel you can still solve the following problem from one of the previous homeworks. Solving it does not count to the restrictions of "at most one problem".

Exercise 16 ($\star$)**.** *Assume that X is a non-empty perfect Polish space and $R \subseteq X^2$ is a comeagre set. Prove that there exists a Cantor set $C \subseteq X$ and a comeagre set $D \subseteq X$ such that $C \times D \subseteq R$.*

As a hint, you may focus on constructing $C \subseteq X$ more than on finding $D \subseteq Y$: begin by finding a Cantor set C such that the set

$$\{y \in Y \mid \forall x \in C.\ (x, y) \in R\}$$

is dense in Y.

5 Hints

Hint to Exercise 3 Notice that $A \in N_s \cap [\mathbb{N}]^{\aleph_0}$ if and only if

$$\forall i < |s|.\ s(n) = 1 \Leftrightarrow i \in A.$$

Thus, one can take $n = |s| - 1$ and $r = \{i \leqslant n \mid s(i) = 1\}$.

Hint to Exercise 4 It is obvious, because the definition is symmetric by taking $[\mathbb{N}]^{\aleph_0} - \mathcal{U}$ instead of $\mathcal{U}$.

Hint to Exercise 5 Both sets are Ramsey for the same reason: take $i = 0$ and $A = \{1\} \times \mathbb{N}$. Then $[A]^{\aleph_0} \cap \mathcal{U}_0 = \varnothing$ and we are done.

Hint to Exercise 6 Assume towards the contradiction that there exists $A \in [\{0, 1\} \times \mathbb{N}]^{\aleph_0}$ such that $[A]^{\aleph_0} \subseteq \mathcal{U}_0 \cup \mathcal{U}_1$ or $[A]^{\aleph_0} \cap (\mathcal{U}_0 \cup \mathcal{U}_1) = \varnothing$.

Since A is infinite, it must contain infinitely many pairs of the form (i, n) for $i = 0$ or for $i = 1$. Thus, without loss of generality assume that $A = \{i\} \times A'$ for $i \in \{0, 1\}$ and some infinite $A' \subseteq \mathbb{N}$ (making A smaller while still infinite does not change anything). Then either $[A]^{\aleph_0} \subseteq \mathcal{U}_i$ or $[A]^{\aleph_0} \cap \mathcal{U}_i = \varnothing$ (the set $\mathcal{U}_{1-i}$ does not play any role in these inclusions). But this implies that $[A']^{\aleph_0} \subseteq \mathcal{U}$ or $[A']^{\aleph_0} \cap \mathcal{U} = \varnothing$, contradicting the fact that $\mathcal{U}$ is not Ramsey.

Hint to Exercise 7 For $s = \varnothing$ and $B = \mathbb{N}$ we get that $\mathcal{U}$ is just Ramsey.

Hint to Exercise 8 Consider any $s < B$. Recall the notion of $\bar{r}$ for $r \in 2^{<\omega}$ from above Exercise 8. Let $n = |r| - 1$. If $\bar{r} \neq s \cap \{0, \dots, n\}$ then $[s, B/n] \cap \mathcal{U} = \varnothing$ (please check it!) and we are done. So assume that $\bar{r} = s \cap \{0, \dots, n\}$. But then $[s, B/n] \subseteq \mathcal{U}$ and we are done again.

What is the role of B/n instead of B above?

Hint to Exercise 10 Observe that the family of completely Ramsey sets (in the same manner as in Exercise 4) is closed under complements. Then use Exercise 9 and the fact that $\mathcal{U}_0 \cap \mathcal{U}_1 = (\mathcal{U}_0^{\mathrm{c}} \cup \mathcal{U}_1^{\mathrm{c}})^{\mathrm{c}}$.

Hint to Exercise 11 Both statements are obvious: if Y accepts s then we use the fact that $[s, Z] \subseteq [s, Y]$, if Y rejects s then we use the fact that $[Z]^{\aleph_0} \subseteq [Y]^{\aleph_0}$.

Hint to Exercise 12 As indicated in lecture notes: if some $X \in [Y]^{\aleph_0}$ accepts s then we take such X. Otherwise, $X = Y$ already rejects s.

Hint to Exercise 13 Apply Exercise 12 inductively, shrinking the set $Y \supseteq X_1 \supseteq X_2 \supseteq \dots \supseteq X_n$, with all X_i infinite in such a way that X_i either accepts or rejects s_i. In the meantime use Exercise 11 to notice that we cannot break the condition that (X_i either accepts or rejects s_i) by taking even smaller sets X_{i+1}, $X_{i+2}, \dots$