Tutorial from 07.05.2020

I am willing to answer your questions and comments.

You can gain points for indicating non-trivial mistakes in the notes!

Homework deadline: **24:00 on Wednesday 13.05**

1 Solutions of the homework problems

Exercise 1. *Construct Borel graphs on 2^ω that have the following properties (one graph for each property):*

1. *countably many connected components,*
2. *continuum many connected components, each of cardinality continuum,*
3. *one connected component but for each n there are $\alpha, \beta \in 2^\omega$ such that the length of the shortest walk from α to β is at least n.*

In each of the above points, argue why the constructed graph is Borel!

Show that if G is a Borel graph on a Polish space X then each connected component of G is an analytic subset of X.

1. Let $(0^n\hat{}1\hat{}\alpha, 0^n\hat{}1\hat{}\beta)$ be connected by an edge if $\alpha \neq \beta$ (by 0^n we denote the sequence of n zeros). The connected components are sets of the form $\{0^n\hat{}1\hat{}\alpha \mid \alpha \in 2^\omega\}$ and a single connected component 0^ω. The graph is a countable union of open sets — open.

2. Let (α, β) be connected by an edge if $\alpha \neq \beta$ and for every n we have $\alpha(2n) = \beta(2n)$. The graph is closed and its connected components are given by the even positions of sequences.

3. Add the following edges between $\alpha \neq \beta$:

   - when $\alpha, \beta \in \mathbb{Q}$ and differ on exactly one position,
   - when $\alpha = 0^\omega$ and $\beta = 1^\omega$,
   - when $\alpha, \beta \notin \mathbb{Q}$.

The graph is clearly Borel because $\mathbb{Q}$ is Borel. Moreover, it has one connected component. However, the shortest walk from 0^ω to $1^n{}^\frown 0^\omega$ has length n (in terms of edges, or $n+1$ in terms of vertices).

Let G be a Borel graph. For $n = 1\ldots$ define C_n as

$$C_n = \big\{(x_0, x_{n+1}) \mid \exists x_1, \ldots, x_n.\ (x_0, x_1), \ldots, (x_n, x_{n+1}) \in G^*\big\}.$$

It is easy to check that each of the sets C_n is analytic. Put $C = \bigcup_{n>0} C_n$ and notice that C is analytic as well. Observe that C is an equivalence relation and its equivalence classes are connected components of G. Each section of C is an analytic set itself (continuous pre-image of an analytic set is analytic).

Exercise 2 ($\star$)**.** *Assume that X is a non-empty perfect Polish space and $R \subseteq X^2$ is a comeagre set. Prove that there exists a Cantor set $C \subseteq X$ and a comeagre set $D \subseteq X$ such that $C \times D \subseteq R$.*

No-one has solved that exercise so it stays open for the next week :)

Notice that there is a certain difficulty here. Consider $X = [0, 1]$ and $R = X^2 - \{(x, x) \mid x \in [0, 1]\}$ dense and open. Notice that there is no open set $C \subseteq X$ and dense G_δ set $D \subseteq X$ such that $C \times D \subseteq R$!

2 New material

First, the standard infinite Ramsey theorem is given.

Theorem 2.1 (Ramsey)**.** *If $P_0 \cup P_1$ is a partition of $[\mathbb{N}]^2$ then there is an infinite set $A \subseteq \mathbb{N}$ that is homogeneous: either $[A]^2$ is a subset of P_0 or of P_1.*

Exercise 3. *Prove the above fact inductively.*

You might construct A inductively, asking for an infinite set $I_k \subseteq \mathbb{N}$ (fix $I_0 = \mathbb{N}$): is there $n_k \in I_k$ such that for infinitely many $m \in I_k$ we have $\{n_k, m\} \in P_0$. If yes, take such n_k and define $I_{k+1} = \{m \in I_k \mid \{n_k, m\} \in P_0\}$. If for some k there is no such n_k then one can easily construct (via another induction) $A \subseteq I_k$ with $[A]^2 \subseteq P_1$.

Entail the following variant from the binary one:

Exercise 4 (Ramsey)**.** *If $P_0 \cup P_1 \cup \ldots \cup P_\ell$ is a partition of $[\mathbb{N}]^2$ then there is an infinite set $A \subseteq \mathbb{N}$ that is homogeneous: $[A]^2 \subseteq P_i$ for some i.*

The example of Sierpiński shows that one cannot directly extend Ramsey's theorem to other cardinalities. Take a fixed well-order $\preceq$ on $\mathbb{R}$. Define a graph $G_{\preceq}$ on $\mathbb{R}$ by putting $(x, y) \in G^*_{\preceq}$ if $x \neq y$ and $\big(x < y \Leftrightarrow x \prec y\big)$.

Exercise 5. *Assume that $X \subseteq \mathbb{R}$ is well-ordered by the standard order $\leqslant$ on $\mathbb{R}$. Prove that $|X| \leqslant \aleph_0$.*

Exercise 6. *Show that the Sierpiński graph $G_{\preceq}$ does not have any uncountable clique nor uncountable anti-clique.*

Galvin's theorem (page 3 of the notes) says that if a graph has Baire Property then there is a Cantor homogeneous set (clique or anti-clique). Its proof is based on Mycielski-Kuratowski but it requires a bit more...It gives a nice corollary about BP-measurable functions (page 6).

Exercise 7. *Prove inductively the more general variant of Galvin's theorem (middle of page 7): if $[X]^2$ is partitioned into $P_0 \cup \ldots \cup P_{n-1}$ with all these sets having Baire Property then there exists a homogeneous Cantor set.*

Another way of extending Galvin's theorem would be to go to triples.

Theorem 2.2 (Ramsey)**.** *If $[\mathbb{N}]^3 = P_0 \cup \ldots \cup P_{n-1}$ is a partition of the three-element subsets of $\mathbb{N}$ then there exists a homogeneous infinite set $A \subseteq \mathbb{N}$ such that $[A]^3 \subseteq P_i$ for some i.*

The next stage of the lecture notes is an example showing that one cannot extend Galvin's result to triples. First, given two $\alpha \neq \beta \in 2^\omega$ one defines $\Delta(\alpha, \beta) = \min\{n \mid \alpha(n) \neq \beta(n)\}$. Then, let P_0 contain a triple $\{\alpha, \beta, \gamma\}$ if $\alpha <_{\text{lex}} \beta <_{\text{lex}} \gamma$ and $\Delta(\alpha, \beta) \leqslant \Delta(\beta, \gamma)$. Dually, P_1 contains such a triple if $\Delta(\alpha, \beta) > \Delta(\beta, \gamma)$.

Recall that if $P \subseteq [X]^n$ then $P^* = \big\{(x_1, \ldots, x_n) \mid \{x_1, \ldots, x_n\} \in P\big\}$.

Exercise 8. *Check that both P_0^* and P_1^* as subsets of $\{(\alpha, \beta, \gamma) \in (2^\omega)^3 \mid \alpha \neq \beta \wedge \beta \neq \gamma \wedge \gamma \neq \alpha\}$ are clopen.*

In the above exercise it is important to remove the diagonal, when speaking about the fact that P_i^* are clopen. This disappears later, when we require them to be open, because the sets like $\{(x, y) \in X^3 \mid x \neq y\}$ are open themselves.

Exercise 9. *Show that if $A \subseteq 2^\omega$ is a 0-homogeneous set for the above partition (i.e. $[A]^2 \subseteq P_0$) then all points except at most one in A are isolated. Entail that A is at most countable.*

Check the same for 1-homogeneous A.

Then the lecture notes move to $OCA^*(A)$ — Open Colouring Axiom⋆ for a set A: if $P_0 \cup P_1$ is a partition of $[A]^2$ such that P_0^* is open in A^2 (as an induced subspace of X^2) then either:

- there exists a 0-homogeneous Cantor set;
- A is a countable union of 1-homogeneous sets.

The presented theorem by Farah and Todorčević asserts that for all sets A that are analytic. In its proof, it is first important to notice the following simple fact.

Exercise 10. *If $OCA^*(A)$ for $A \subseteq X$ holds and $f\colon X \to Y$ is a continuous function between Polish spaces then $OCA^*(f(A))$ holds as well.*

Thus, instead of showing $OCA^*(A)$ one can show $OCA^*(X)$ with X being some Polish space whose continuous image is A.

The rest of the proof is an analogue of the Cantor-Bendixson theorem: instead of the σ-ideal of countable sets, we work with the σ-ideal of countable unions of 1-homogeneous sets.

Exercise 11. *Let $\mathcal{I}$ be a σ-ideal on a Polish space X, assume that $X \notin \mathcal{I}$ ($\mathcal{I}$ is not full) and $[X]^{<\aleph_0} \subseteq \mathcal{I}$ — all finite subsets of X belong to $\mathcal{I}$. Then one can partition $X = Y \cup U$ with a perfect (no isolated points) set $Y \notin \mathcal{I}$ and an open set $U \in \mathcal{I}$. Moreover, if V is an open subset of X and $V \cap Y \neq \emptyset$ then $V \cap Y \notin \mathcal{I}$ ($V \cap Y$ is a* relatively open *subset of Y).*

Hint from the lecture notes: put $U = \bigcup\{U_n \mid U_n \in \mathcal{I}\}$ for some countable basis $(U_n)_{n\in\mathbb{N}}$ of the topology of X.

This allows us to conclude the proof of the theorem by Farah and Todorčević, see page 12.

Based on that, we can have the following strengthening of the perfect set property for analytic sets.

Proposition 2.3. *Assume that $f\colon X \to Y$ is a Borel function between Polish spaces X and Y. If the range of f is uncountable then there is a Cantor set $C \subseteq X$ such that $f\restriction_C$ is injective.*

Take as A the graph of f, i.e. $A = \{(x, y) \mid f(x) = y\}$ (our basic space is $X \times Y$). Clearly, A is Borel and therefore analytic. Consider the partition of $[A]^2 = P_0 \cup P_1$ with P_0 containing $\{(x, y), (x', y')\}$ if $y \neq y'$. It is clear that P_0^* is open in A^2 so it is an open partition. Apply $OCA^*(A)$ to that partition.

Exercise 12. *Assume that there exists a* 0*-homogeneous Cantor set. Entail that there is a Cantor set* $C \subseteq X$ *such that* $f{\restriction}_C$ *is injective.*

Exercise 13. *Assume that* A *is a countable union of* 1*-homogeneous sets. Entail that the range of* f *is countable.*

Then yet another application is given, see the theorem by von Douwen and Przymusiński on page 14. In its proof, we have a Polish space X and an analytic set $A \subseteq X^2$. We partition $[A]^2$ (again this is essentially a subspace of X^4 (!)) into $P_0 \cup P_1$ with P_0 containing $\{(x_1, x_2), (y_1, y_2)\}$ if $(x_1, x_2), (y_1, y_2) \in A$ and moreover $\{x_1, x_2\} \cap \{y_1, y_2\} = \varnothing$.

Exercise 14. *Check that* P_0^* *is open in* A^2 *with the topology induced from* X^4*.*

Exercise 15. *Assume that* A *is a countable union of* 1*-homogeneous sets. Show that* A *can be covered by a countable union of* lines *— sets of the from* $X \times \{x_0\}$ *or* $\{x_0\} \times X$*.*

3 New homework

This time you are allowed to solve both problems if you like :)

The first exercise should be doable by invoking some known principles from the lecture — please invoke them precisely and write down carefully all the remaining reasoning.

Exercise 16. *Prove that if* $B \subseteq X \times Y$ *is Borel and all its sections* B_x *for* $x \in X$ *are at most countable then the projection* $\pi_X(B)$ *is Borel itself.*

Exercise 17 ($\star$)**.** *Show that there exists a partition of* $[2^\omega]^2$ *into infinitely many pieces* $(P_i)_{i \in \mathbb{N}}$ *such that the pieces are clopen (i.e.* P_i^* *is clopen in* $\{(x, y) \in 2^\omega \times 2^\omega \mid x \neq y\}$*) and this partition does not admit any homogeneous copy of the Cantor set.*

4 Old homework

In parallel you can still solve the following problem from the previous homework.

Exercise 18 ($\star$)**.** *Assume that X is a non-empty perfect Polish space and $R \subseteq X^2$ is a comeagre set. Prove that there exists a Cantor set $C \subseteq X$ and a comeagre set $D \subseteq X$ such that $C \times D \subseteq R$.*

5 Hints

Hint to Exercise 3 Take $I_0 = \mathbb{N}$ and inductively proceed as suggested. If for all $k = 0, \ldots$ there is a respective number n_k then the set $A = \{n_k \mid k \in \mathbb{N}\}$ is 0-homogeneous. Assume that at some stage k the respective number n_k does not exist. I.e. the infinite set $J_0 = I_k$ has the property that for every $n \in J_0$ there is only finitely many $m \in J_0$ with $\{n, m\} \in P_0$. Proceed inductively for $\ell = 0, \ldots$ taking $n_\ell = \min J_\ell$ and $J_{\ell+1} = \big\{m \in J_\ell \mid \{n_\ell, m\} \notin P_0\big\}$. Then the sets J_ℓ are all infinite and the construction of $(n_\ell)_{\ell \in \mathbb{N}}$ must succeed.

Hint to Exercise 4 Induction on ℓ. For $\ell = 0$ trivial, for $\ell = 2$ the basic Ramsey. Assume the thesis for ℓ, consider $P_0 \cup \ldots P_\ell$ and treat it as $P_0, \ldots, (P_{\ell-1} \cup P_\ell)$. Apply the previous result and if the set A is homogeneous for $P_{\ell-1} \cup P_\ell$ apply the standard Ramsey once again for the infinite set A and the partition $P_{\ell-1} \cup P_\ell$.

Hint to Exercise 5 Notice that for easy $x \in X$ there exists a rational number q_x such that $x = \max\{y \in X \mid y \leqslant q_x\}$: if $x = \max X$ then take any rational greater than x, otherwise take $x' = \min\{y \in X \mid y > x\}$ and any rational in $[x, x')$. Thus, the function $x \mapsto q_x$ is an injection of X into rationals $\mathbb{Q}$.

Hint to Exercise 6 First take any clique X in $G_\preceq$. By the definition of $G_\preceq$ it means that the orders $\leqslant$ and $\preceq$ coincide on X. Therefore, X is well-ordered by $\leqslant$. Use Exercise 5.

In the case when X is an anti-clique, the orders $\geqslant$ and $\preceq$ coincide on X, so we can apply Exercise 5 to $-X = \{-x \mid x \in X\}$.

Hint to Exercise 7 The same induction as in Exercise 4 — in the case when $[C]^2 \subseteq P_{\ell-1} \cup P_\ell$ apply standard Galvin again to the perfect Polish space C.

Hint to Exercise 8 Notice that if $(\alpha, \beta, \gamma) \in P_0^*$ then it is witnessed by some finite prefixes $(\alpha\restriction_n, \beta\restriction_n, \gamma\restriction_n)$. Therefore, P_0^* is open. The same holds for P_1^*. However, $P_0^* \cup P_1^*$ is the whole set

$$\{(\alpha, \beta, \gamma) \in (2^\omega)^3 \mid \alpha \neq \beta \wedge \beta \neq \gamma \wedge \gamma \neq \alpha\}.$$

Hint to Exercise 9 Take any non-$\leqslant_{\text{lex}}$-maximal point α in A and let $\gamma >_{\text{lex}} \alpha$ witness that. Let $\Delta(\alpha, \gamma) = n$. Assume for the sake of contradiction that there exists $\beta \neq \alpha \in A$ such that $\Delta(\alpha, \beta) > n$. Notice that it implies that $\beta <_{\text{lex}} \gamma$ and $\Delta(\beta, \gamma) = n$. Thus, no matter whether $\alpha <_{\text{lex}} \beta$ or $\beta <_{\text{lex}} \alpha$ we have a contradiction with the fact that $\{\alpha, \beta, \gamma\} \in P_0$.

The case of P_1 is analogous, take any non-$\leqslant_{\text{lex}}$-minimal point γ in A.

Now, if a set A consists of purely isolated points then there is at most as many points in A as basic open sets of our space. So $|A| \leqslant \aleph_0$.

Hint to Exercise 10 Take $P_0 \cup P_1 = [f(A)]^2$ with P_0^* open in $f(A)^2$. Take R_0 containing $\{x, y\}$ if P_0 contains $\{f(x), f(y)\}$. Let R_1 be the remainder of $[A]^2$. Then R_0^* is open in A^2. Apply $OCA^*(A)$ to the partition $R_0 \cup R_1 = [A]^2$.

Consider the first case that there exists a 0-homogeneous Cantor set C in A, i.e. $[C]^2 \subseteq R_0$. Notice that f must be injective on C, because if $\{x, y\} \in R_0$ then not only $x \neq y$ but also $f(x) \neq f(y)$. Therefore, $f(C)$ is a 0-homogeneous Cantor set (homogeneous for P_0) — injectivity of f is important here, because if $f(C)$ was a singleton, we'd be left with nothing...

Now assume that A is a countable union of 1-homogeneous sets $(F_n)_{n\in\mathbb{N}}$. Consider the sets $f(F_n)$. Observe that if $x \neq y \in f(F_n)$ would satisfy $\{x, y\} \in P_0$ then some two points in F_n would be connected by R_0, contradicting the assumption that F_n is 1-homogeneous. Therefore, $f(F_n)$ is 1-homogeneous (homogeneous for P_1). Thus, $f(A)$ is a countable union of 1-homogeneous sets $f(F_n)$.

Hint to Exercise 11 Using the hint to take $U = \bigcup\{U_n \mid U_n \in \mathcal{I}\}$ for some countable basis $(U_n)_{n\in\mathbb{N}}$ of the topology of X we see that $U \in \mathcal{I}$. Take $Y = X - U$.

If Y contained any isolated point $\{y_0\} = Y \cap U_n$ for some n then $U_n \in \mathcal{I}$ as a subset of the union of $U \in \mathcal{I}$ and $\{y_0\} \in \mathcal{I}$. Therefore, Y has no isolated points. Clearly, if we had $Y \in \mathcal{I}$ then $X = Y \cup U \in \mathcal{I}$ while we assumed that $X \notin \mathcal{I}$.

Take any open subset V of X such that $V \cap Y \neq \emptyset$. We aim at showing that $V \cap Y \notin \mathcal{I}$. Let $U_n \subseteq V$ be any basic set such that $U_n \cap Y \neq \emptyset$. It is enough to show that $U_n \cap Y \notin \mathcal{I}$, as $U_n \cap Y \subseteq V \cap Y$. But if $U_n \cap Y \in \mathcal{I}$ then $U_n \in \mathcal{I}$ because U also belongs to $\mathcal{I}$, and then $U_n \cap Y = \emptyset$.

Hint to Exercise 12 Take a 0-homogeneous cantor set $C \subseteq X \times Y$. By the definition of P_0 we know that if $(x, y), (x', y') \in C$ are two distinct elements of C then $y \neq y'$. As A is the graph of f it means that also $x \neq x'$. Therefore, the projection of C onto X is also a Cantor set (the projection operation is injective on C). We claim that this projection (denoted C') has the property that $f\restriction_{C'}$ is injective. However, if $x \neq x' \in C'$ then $(x, f(x)), (x', f(x')) \in C$ because $C \subseteq A$ and therefore by the choice of P_0 we have $f(x) \neq f(x')$.

Hint to Exercise 13 Each 1-homogeneous set must contain points (x, y) of fixed second coordinate y.

Hint to Exercise 14 It is enough to notice that the condition $\{(x, y) \in X^2 \mid x \neq y\}$ is open and then take some finite intersections.

Hint to Exercise 15 It is enough to show that if F is a 1-homogeneous set then F is contained in a finite union sets of the form $X \times \{x_0\}$ or $\{x_0\} \times X$.

Take any $(x, y) \in F$ and observe that

$$F \subseteq \{x\} \times X \cup \{y\} \times X \cup X \times \{x\} \cup X \times \{y\}.$$