Tutorial from 30.04.2020

I am willing to answer your questions and comments.

You can gain points for indicating non-trivial mistakes in the notes!

Homework deadline: **24:00 on Thursday 07.05**

1 Solutions of the homework problems

Exercise 1. *Let $X_1, \ldots, X_n$ be non-empty perfect Polish spaces. If A is comeagre in $\prod_{i \leqslant n} X_i$ then there exist copies of the Cantor set $C_1, \ldots, C_n$ in $X_1, \ldots, X_n$ respectively such that $\prod_{i \leqslant n} C_i \subseteq A$.*

Let X be a non-empty perfect Polish space and $n \geqslant 1$. If $A \in \mathrm{BP}(X^n) - \mathrm{MGR}(X^n)$ (i.e. A has Baire Property but is not meagre) then there exist copies of the Cantor set $C_1, \ldots, C_n$ in X such that $\prod_{i \leqslant n} C_i \subseteq A$.

This argument is based on a solution by Anna Balicka.

Consider the first part of the statement. W.l.o.g. assume that A is a dense G_δ being an intersection of a descending family of open sets U_n.

It is enough to repeat the construction from the lecture, this time constructing n Cantor schemes V^i_s for $i = 1, \ldots, n$ and $s \in 2^{<\omega}$. They need to satisfy the standard conditions on monotonicity (if $s \prec r$ then $\overline{V^i_r} \subseteq V^i_s$), diameters $\mathrm{diam}(V^i_s) \leqslant 2^{-|s|}$), and disjointness (if $s \neq r \in 2^n$ then $\overline{V^i_s} \cap \overline{V^i_r} = \varnothing$). Moreover, we ensure that for each choice of $s_1, \ldots, s_n \in 2^n$ we have $V^1_{s_1} \times \ldots \times V^n_{s_n} \subseteq U_n \subseteq A$. The inductive step of defining $(V_s)_{s \in 2^{n+1}}$ assuming that we have $(V_s)_{s \in 2^n}$ is even simpler than in the lecture notes because by density of U_{n+1} we know that there is some basic open set there — i.e. a product of some basic sets in X.

The second part follows from the first one, because under the given assumptions, A is comeagre in some basic open set $U_1 \times \ldots \times U_n$ of X^n and we an invoke the first part for the perfect Polish spaces $X_i = U_i$ for $i = 1, \ldots, n$.

Exercise 2 ($\star$)**.** *Show that there exists a copy of the Cantor set $C \subseteq \mathbb{R}$ whose members are linearly independent over $\mathbb{Q}$: if $x_1, \ldots, x_n \in C$ are pairwise distinct and $\sum_{i \leqslant n} q_i \cdot x_i = 0$ for some rational numbers $q_1, \ldots, q_n \in \mathbb{Q}$ then all the numbers q_i equal 0.*

This argument is based on solutions by Damian Głodkowski and Tomasz Przybyłowski.

For each n consider the set $H_n = \mathbb{Q}^n - \{(0,\dots,0)\}$. Notice that H_n is countable and for each $\vec{q} \in H_n$ the set $F_{\vec{q}} = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \vec{q}_i \cdot x_i = 0\}$ is meagre in $\mathbb{R}^n$. Therefore, $F_n = \bigcup_{\vec{q}\in H_n} F_{\vec{q}}$ is also meagre in $\mathbb{R}^n$. Take $A_n = \mathbb{R}^n - F_n$, and put $n_n = n$. Apply the stronger variant of Mycielski–Kuratowski Theorem (page 16 of lecture notes WZMT-8, see Exercise 3 below) to the sequence $A_n \subseteq \mathbb{R}^{n_n}$. The obtained Cantor set C is easily the desired set independent over $\mathbb{Q}$.

2 New material

2.1 Continuation of the previous topic

We will now prove the more general version of the theorem by Mycielski–Kuratowski (it is stated on page 16 of the lecture notes WZTM-8).

Exercise 3. *Let X be a non-empty perfect Polish space. For every sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ and a sequence of sets $(A_k)_{k\in\mathbb{N}}$ such that A_k is comeagre in X^{n_k}, there exists a homeomorphic copy of the Cantor set $C \subseteq X$ such that for every k we have*

$$\left\{\vec{x} \in C^{n_k} \mid \forall i \neq j.\ x_i \neq x_j\right\} \subseteq A_k.$$

Idea (spoiler alert!): repeat the previous construction: w.l.o.g. assume that each A_k is a dense G_δ, i.e. $A_k = \bigcap_{i\in\mathbb{N}} U_{k,i}$ for $U_{k,i}$ open and dense. Construct inductively a Cantor scheme, making sure that the respective products are contained in more and more of the sets $U_{k,i}$.

2.2 2^ω as a topological group

The set $2 = \{0,1\}$ has the natural structure of the additive group $\mathbb{Z}_2$ (sometimes addition is also called XOR). This operation can be extended to 2^ω in the coordinate-wise way.

Exercise 4. *Prove that addition is continuous as a function $2^\omega \times 2^\omega \to 2^\omega$.*

Notice that the sequence constantly equal 0 is the neutral element of this addition.

Exercise 5. *Is there an inverse operation in 2^ω: given $\alpha \in 2^\omega$ can one find α^{-1} such that $\alpha + \alpha^{-1} = \alpha^{-1} + \alpha = 0$.*

Therefore, we know that 2^ω is a topological group: both addition and inverse are continuous functions.

Now, the lecture notes denote the set of sequences ultimately equal 0 by $\mathbb{Q}$ and call them *rationals*.

Exercise 6. *Show that $\mathbb{Q}$ is a subgroup of 2^ω.*

This leads to the idea of a quotient: we call two elements $\alpha, \beta \in 2^\omega$ equivalent (denoted $\alpha \sim \beta$) if $\alpha - \beta \in \mathbb{Q}$. The Vitali set S constructed in lecture notes is just a selector of the layers of the quotient — it contains a single element from each equivalence class of $\sim$.

If $\alpha \in 2^\omega$ and $A \subseteq 2^\omega$ then by $\alpha + A$ we denote the set $\{\alpha + \beta \mid \beta \in A\}$.

Exercise 7. *Notice that if S is a Vitali set then $\{q+S\}_{q\in\mathbb{Q}}$ is a split of 2^ω into countably many homoemorphic pairwise disjoint subsets. (it was previously claimed here that the sets are dense, which in general doesn't have to be the case)*

2.3 Graphs

This lecture is devoted to descriptive graph theory. We view a graph as the set of its oriented edges $G^* \subseteq V(G)^2$ and study the topological complexity of that set (assuming $V(G)$ to be some nice Polish space, like 2^ω). The only two requirements about G^* are that it is symmetric (if $(x, y) \in G^*$ then $(y, x) \in G^*$) and anti-reflexive ($(x, x) \notin G^*$) [this last condition is not entirely kosher, some graph theorists would allow self-loops but in the end it does not matter much].

Then the standard graph theoretic notions of a walk, clique, anti-clique, and colouring by some set Z are defined. Homomorphisms of graphs are defined as for relational structures: the image of an edge must be an edge but some new edges may arise in the range of the homomorphism.

Then on page 4 a very special graph G_0 on 2^ω is constructed. Notice that the actual structure of that graph depends on the choice of the sequences s_n, thus G_0 is in fact a family of graphs. On the other hand, it is possible to inductively construct some sequence s_n satisfying the requirements (see page 5) so one should think about G_0 as a *constructive* object — contrary to the set X from the previous exercises or the related Vitali set from Exercise 7.

Exercise 8. *Check that* $G_0 \subseteq 2^\omega \times 2^\omega$ *is* F_σ.

Exercise 9. *Show that* G_0 *has no isolated points.*

Then proposition at the end of page 6 shows, that the connected components of G_0 are precisely the equivalence classes of the relation $\sim$ from the end of Subsection 2.2. The proof of this result is not entirely trivial, because we can only swap 0 with 1 (or 1 with 0) at the end of a sequence s_n for some $n \in \mathbb{N}$. Thus, some inductive construction is needed to show how to construct a (possibly very long) walk from α to β under the assumption that $\alpha \sim \beta$.

Exercise 10. *Assume that* α *differs from* β *on* N *positions* $\{n_0, \ldots, n_N\}$. *Can we give some upper bound on the length of the shortest walk from* α *to* β *in* G_0*?*

We know that that each connected component of G_0 is an equivalence class of $\sim$. Moreover, each edge of G_0 swaps one position of the given word. Thus, we can go chess-like: some vertices are white, others are black, every two neighbours have distinct colours, so every loop has even length. This leads to the following realisation from page 9.

Exercise 11. *The graph* G_0 *is* 2*-colourable.*

Similarly as in the exercise about a set X that is sensitive to changing single bits, no 2-colouring of G_0 can have Baire property (seen as a set of vertices of colour 0 and the complement coloured 1).

2.4 Lusin–Novikov theorem

The next step is an introduction (luckily for us without a proof) of the celebrated result by Kechris, Solecki, and Todorčević.

Theorem 2.1 (Kechris–Solecki–Todorčević)**.** *If* G *is an analytic graph on a Polish space* X *then either it has a Borel* $\mathbb{N}$*-colouring or one can continuously embed* G_0 *into* G.

Notice that the definition of a Z-colouring does not depend on the actual structure of Z, only on its cardinality, so we could say that the first possibility above says that G admits any $\aleph_0$-colouring.

The first application of this result is Lusin-Novikov theorem (that can also be proved more directly), see page 13 of the notes.

Theorem 2.2 (Lusin–Novikov)**.** *Take $B \subseteq X \times Y$ that is Borel. Then either B is a union of graphs of countably many Borel partial functions $X \rightharpoonup Y$; or B has an uncountable vertical section.*

Exercise 12. *Show that if $B \subseteq X \times Y$ is Borel and has an uncountable section then there is a section of B that contains a copy of the Cantor set.*

To prove the above theorem, a graph G on $X \times Y$ (i.e. $G \subseteq (X \times Y)^2$) is defined:

$$G = \left\{ \big((x, y), (x', y')\big) \mid x = x' \wedge y \neq y' \wedge (x, y) \in B \wedge (x', y') \in B \right\}.$$

In other words, the edges of G allow to move within a vertical section of B between one point and another. However, the strongly connected components of G are all contained in single vertical sections of B (i.e. the x coordinate is constant).

Exercise 13. *Show that G is a Borel graph (i.e. it is just a Borel subset of $X \times Y \times X \times Y$).*

Since all Borel sets are analytic, one can apply KST and study the two possibilities. The first case is almost trivial, while the second requires a bit more work: we find a candidate for a section B_{x_0} for some $x_0 \in X$ and argue that this section cannot be countable, because then G_0 would be a union of countably many **closed** anti-cliques.

2.5 Silver's theorem

The next (and final) application of KST is Silver's theorem on coanalytic equivalence relations. Please carefully study its proof, because it nicely embraces the results that we have seen so far!

Silver's theorem easily implies the following claim (that we already know).

Exercise 14. *Let $A \subseteq X$ be an uncountable analytic subset of a Polish space X. Then A contains a copy of the Cantor set.*

Can you deduce the above result directly from KST's Theorem?

3 New homework

Please choose at most one exercise this time. You can solve the first one partially — I will grant some points for the separate items there. Most probably the whole exercise will be worth more than 1 point.

Exercise 15. *Construct Borel graphs on 2^ω that have the following properties (one graph for each property):*

1. *countably many connected components,*
2. *continuum many connected components, each of cardinality continuum,*
3. *one connected component but for each n there are $\alpha, \beta \in 2^\omega$ such that the length of the shortest walk from α to β is at least n.*

In each of the above points, argue why the constructed graph is Borel!

Show that if G is a Borel graph on a Polish space X then each connected component of G is an analytic subset of X.

Exercise 16 ($\star$)**.** *Assume that X is a non-empty perfect Polish space and $R \subseteq X^2$ is a comeagre set. Prove that there exists a Cantor set $C \subseteq X$ and a comeagre set $D \subseteq X$ such that $C \times D \subseteq R$.*

4 Hints

Hint to Exercise 3 I hope that the idea given in the main body is enough. If anything is not clear here, please write to me directly and we can discuss it during next Zoom office hours.

Hint to Exercise 4 One way is to notice that it is continuous when composed with projections: the function $2^\omega \times 2^\omega \to 2$ that returns the nth bit of $\alpha + \beta$ is continuous.

Hint to Exercise 5 In fact $\alpha + \alpha = 0$ for each α, so the inverse is the identity function (that is continuous).

Hint to Exercise 6 I hope that it is clear: if α and β have almost only 0's then $\alpha + \beta$ also has almost only 0's. The inverse is identity so nothing to check here.

Hint to Exercise 7 First, if $\alpha \in 2^\omega$ then S contains a unique member α' of the $\sim$ equivalence class of α. Therefore, $q = \alpha' - \alpha$ has the property that $\alpha \in q + S$.

Assume that $\alpha \in q + S \cap q' + S$, as witnessed by $\beta, \beta' \in S$ such that $q + \beta = q' + \beta' = \alpha$. But then $\beta \sim \beta'$ because $\beta - \beta' = q' - q \in \mathbb{Q}$ and by the choice of S we know that $\beta = \beta'$. But then $q = q'$.

Hint to Exercise 8 G_0 can be written as the union ranging for $n \in \mathbb{N}$ of the set

$$\{(s_n \frown 0 \frown x, s_n \frown 1 \frown x) \mid x \in 2^\omega\}, \tag{4.1}$$

and its reverse (swap 0 and 1) to make the relation symmetric. Thus, it is enough to show that the set in (4.1) is closed. But it is the intersection of the sets requiring that the first n positions equal s_n on both coordinates; that the n+1th positions are 0 and 1; and that each of the remaining positions is equal on both coordinates. All those sets are basic in $2^\omega \times 2^\omega$ so their intersection is closed. Please check that carefully if you feel unsure about that argument.

Hint to Exercise 9 Every point $\alpha \in 2^\omega$ has an edge to α' defined as α with swapped first position, because $s_n = <>$.

Hint to Exercise 10

Hint to Exercise 11 The formula for the colouring is given in the lecture notes. One easily checks that it is in fact a 2-colouring (i.e. each edge connects two vertices of distinct colours) similarly as in the set X from the previous exercises. Again, contact me if it isn't clear!

Hint to Exercise 12 If $B \subseteq X \times Y$ is Borel then each of its sections B_{x_0} is Borel in Y: take a function $y \mapsto (x_0, y)$ and take the preimage of B — preimages of Borel sets are Borel, because it is the case for closed sets, and preimages go well with (countable) Boolean operations. What does it exactly mean to "go well"? :)

Hint to Exercise 13 The definition of G is just an intersection of two copies of B (on the first two coordinates and on the last two coordinates) and the set of (x, y, x', y') such that $y \neq y'$ — this set is open.

Hint to Exercise 14 Consider the set $E \subseteq X \times X$ of pairs (x, y) such that either $x, y \notin A$ or $x = y$. Check that since A is analytic, E is coanalytic. Check that E is an equivalence relation. Apply Silver's theorem. Notice that the equivalence classes of E are of two kinds: either $\{x\}$ for $x \in A$ and a single class $X - A$.

If E has countably many equivalence classes then A is countable.

If there is a copy of Cantor set C that is E independent, then at most one point of C belongs to $X - A$ and all the other points belong to A. We can now cut C into two sub-Cantors and one of them must be fully contained in A.