Tutorial from 23.04.2020

I'm really willing to answer your questions and comments.

I will grant additional points if someone indicates any "not-entirely-trivial" mistake in the notes!

Homework deadline: **24:00 on Wednesday 29.04**

1 Solutions of the homework problems

Exercise 1 ($\star$)**.** *Let $X \subseteq 2^\omega$ be a set such that for every $\alpha \in 2^\omega$ and every $n \in \mathbb{N}$ we have*

$$\alpha \in X \Leftrightarrow \alpha' \notin X,$$

where α' is the same as α except that $\alpha'(n) = 1 - \alpha(n)$. In other words, for every two sequences $\alpha, \alpha' \in 2^\omega$ that differ on **exactly** *one position, X contains exactly one of them. Prove that such a set exists using the axiom of choice. Prove that every set X satisfying the above condition does not have Baire Property.*

Consider a relation on 2^ω such that $\alpha \sim \beta$ iff α differs from β on finitely many positions. Clearly $\sim$ is an equivalence relation. Let X_0 be a selector of $\sim$ that contains a single point from each equivalence class of $\sim$. Notice that for each $\alpha \in 2^\omega$ there is a unique $\alpha' \in X_0$ such that $\alpha \sim \alpha'$. Define X as containing those points α such that the respective $\alpha' \in X_0$ satisfying $\alpha \sim \alpha'$ differs from α on an even number of positions. It is easy to check that X has the claimed properties.

Assume for the sake of contradiction that X has Baire Property. In that case also $2^\omega - X$ must have Baire Property, so without loss of generality we can assume that X is meagre in some basic set $N_s \subseteq 2^\omega$. From that moment on we restrict our attention to N_s and $X \cap N_s$ — in other words, we assume that $s = \langle\rangle$.

Let n be any position and let $f: 2^\omega \to 2^\omega$ be the function that swaps the position number 0 of each sequence, i.e. $f(\alpha)(0) = 1 - \alpha(0)$ and for $n > 0$ we have $f(\alpha)(n) = \alpha(n)$. Notice that f is a homeomorphism of 2^ω and $f \circ f = \mathrm{id}_{2^\omega}$. Let $X' = f(X)$. Since X is meagre and f is a homeomorphism, X' is also meagre. Notice that $2^\omega = X \cup X'$, because each sequence $\alpha \in 2^\omega$ either belongs to X or does not belong to X but then $f(\alpha)$ belongs to X. Contradiction, because 2^ω cannot be a union of two meagre sets.

Exercise 2 ($\star$)**.** *Let $\sigma(\mathbf{\Sigma}^1_1)$ be the σ-algebra generated by analytic sets $\mathbf{\Sigma}^1_1$. Let $\mathcal{A}(\mathbf{\Pi}^1_1)$ be the family of sets obtained via the Souslin operation applied to coanalytic sets. Prove that $\sigma(\mathbf{\Sigma}^1_1) \subsetneq \mathcal{A}(\mathbf{\Pi}^1_1)$ (two tasks: prove the inclusion and prove that it is strict).*

First take $S = \mathcal{A}(\mathbf{\Pi}^1_1) \cap \left[\mathcal{A}(\mathbf{\Pi}^1_1)\right]^{\mathrm{c}}$. Observe that both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are contained in S. Also, S is closed under countable unions and countable intersections, because the operation $\mathcal{A}$ has these properties. Therefore, S is a σ-algebra. Thus, $\sigma(\mathbf{\Sigma}^1_1) \subseteq S$.

To prove that the inclusion is strict, we will show that $\sigma(\mathbf{\Sigma}^1_1(\mathbb{N}^\omega)) \neq S(\mathbb{N}^\omega)$. To achieve that, we use Exercise 7 from homework: we need to construct a universal set for $\mathcal{A}(\mathbf{\Pi}^1_1)$ and notice that $\mathcal{A}(\mathbf{\Pi}^1_1)$ is a boldface pointclass. Then $\mathcal{A}(\mathbf{\Pi}^1_1) \neq \mathcal{A}(\mathbf{\Pi}^1_1)^{\mathrm{c}}$ while $S = S^{\mathrm{c}}$.

First, notice that if $f\colon X \to Y$ is a continuous function and $B \in \mathcal{A}(\mathbf{\Pi}^1_1(Y)$ witnessed by $B = \mathcal{A}(B_s)$ then $f^{-1}(B) \in \mathcal{A}(\mathbf{\Pi}^1_1(X)$ because one can take the scheme $\left(f^{-1}(B_s)\right)_{s\in\mathbb{N}^{<\omega}}$ and use the fact that $\mathbf{\Pi}^1_1$ is itself a boldface pointclass.

We will construct a $(\mathbb{N}^\omega)^{\mathbb{N}^{<\omega}}$-universal set for $\mathcal{A}(\mathbf{\Pi}^1_1(\mathbb{N}^\omega))$ and then use the fact that $(\mathbb{N}^\omega)^{\mathbb{N}^{<\omega}}$ is homeomorphic with $\mathbb{N}^\omega$. Fix a $\mathbb{N}^\omega$-universal set U_Π for $\mathbf{\Pi}^1_1(\mathbb{N}^\omega)$. Consider $U_{\mathcal{A}\Pi}$ defined as:

$$U_{\mathcal{A}\Pi} = \left\{(\alpha, x) \in (\mathbb{N}^\omega)^{\mathbb{N}^{<\omega}} \times \mathbb{N}^\omega \mid \exists \eta \in \mathbb{N}^\omega . \forall n \in \mathbb{N}.\ (\alpha(\eta\restriction_n), x) \in U_\Pi\right\}.$$

First notice that $U_{\mathcal{A}\Pi} \in \mathcal{A}(\mathbf{\Pi}^1_1)$ because for each $s \in \mathbb{N}^{<\omega}$ the set

$$U_s \stackrel{\text{def}}{=} \{(\alpha, x) \in (\mathbb{N}^\omega)^{\mathbb{N}^{<\omega}} \times \mathbb{N}^\omega \mid (\alpha(s), x) \in U_\Pi\}$$

is coanalytic and $U_{\mathcal{A}\Pi} = \mathcal{A}(U_s)$. Now, if $A \in \mathcal{A}(\mathbf{\Pi}^1_1)$ is obtained as $A = \mathcal{A}(A_s)$ for some $A_s \in \mathbf{\Pi}^1_1$ then there exist α_s such that $A_s = \left(U_\Pi\right)_{\alpha_s}$ by universality of U_Π. Let $\alpha = \left(\alpha_s\right)_{s\in\mathbb{N}^{<\omega}}$ be a member of $(\mathbb{N}^\omega)^{\mathbb{N}^{<\omega}}$. It is now easy to check that $A = \left(U_{\mathcal{A}\Pi}\right)_\alpha$.

2 Additional remark

I was asked to provide a proper argument for a previous exercise.

Exercise 3. *Take any set X and any family of sets $\Gamma \subseteq \mathsf{P}(X)$. Assume that Γ contains $\varnothing$ and X. Show that $\mathcal{A}\mathcal{A}\Gamma = \mathcal{A}\Gamma$.*

This proof is based on the argument in Kechris, Proposition 25.6.

Clearly it is enough to show that $\mathcal{A}\mathcal{A}\Gamma \subseteq \mathcal{A}\Gamma$. Let $A = \mathcal{A}(s \mapsto P_s)$ with $P_s \in \mathcal{A}\Gamma$, i.e. $P_s = \mathcal{A}(t \mapsto Q_{s,t})$ (to avoid confusion, by $\mathcal{A}(r \mapsto A_r)$ I denote the Souslin operation on the parameter r). It is easy to check that

$$x \in A \iff \exists \alpha \in \mathbb{N}^\omega.\ \exists \beta \in (\mathbb{N}^\omega)^\omega.\ \forall m.\ \forall n.\ x \in Q_{\alpha\restriction_m, z(m)\restriction_n}.$$

Fix a bijection $\iota\colon \mathbb{N}^2 \to \mathbb{N}$ such that $m \leqslant \iota(m,n)$ and $p < n \Rightarrow \iota(m,p) < \iota(m,n)$ (the zig-zag function from page 6 of tutorial_3_26.pdf is good). Let ι_0^{-1} and ι_1^{-1} be the respective coordinates of the reverse function, i.e. for every $k \in \mathbb{N}$ we have $\iota\big(\iota_0^{-1}(k), \iota_1^{-1}(k)\big) = k$.

Our aim is to encode witnesses $(\alpha,\beta) \in \mathbb{N}^\omega \times (\mathbb{N}^\omega)^\omega$ as single sequences in $\mathbb{N}^\omega$ using the above function shuffling the coordinates. We will encode $(\alpha,\beta) \in \mathbb{N}^\omega \times (\mathbb{N}^\omega)^\omega$ by $w \in \mathbb{N}^\omega$ defined as

$$w(k) \stackrel{\text{def}}{=} \iota\Big(\alpha(k), \beta(\iota_0^{-1}(k))(\iota_1^{-1}(k))\Big).$$

Notice a tiny difference with the previous approach: we not only mix the coordinates using ι but also mix the actual values: a single number (coordinate) in w codes a coordinate of α together with a coordinate of one of the sequences β. This gives the desired bijection.

Note that if we know $w\restriction_{\iota(m,n)}$ then we can determine $\alpha\restriction_m$ (because each coordinate of α goes into w via ι) and also $\beta(m)\restriction_n$ (because the function ι is sufficiently monotone). This gives rise to a pair of functions $\varphi, \psi\colon \mathbb{N}^{<\omega} \to \mathbb{N}^{<\omega}$ such that if w encodes (α,β) in the above sense and $s = w\restriction_{\iota(m,n)}$ then $\varphi(s) = \alpha\restriction_m$ and $\psi(s) = \beta(m)\restriction_n$ (notice that the length of s determines the values of m and n).

Put $R_s = Q_{\varphi(s),\psi(s)}$ and notice that

$$x \in A \Leftrightarrow x \in \mathcal{A}(s \mapsto R_s).$$

3 New material

The new lecture provides tools for showing that sets with Baire Property are *well-behaved*. The first main result, Kuratowski-Ulam theorem is a category-based analogue of the Fubini theorem:

Theorem 3.1 (Fubini)**.** *If $A \subseteq X \times Y$ is a measurable set then almost all sections of A are measurable in X and Y respectively. Moreover, A has measure 0 if and only if almost all its sections have measure 0.*

[[there is also the part about commuting integrals but that piece is boring]]

Exercise 4 (Corollary from page 7 of the lecture notes)**.** *Let X and Y be Polish spaces, $A \subseteq X$ and $B \subseteq Y$. Then:*

- $A \times B \in \mathrm{MGR}(X \times Y)$
  if and only if
  ($A \in \mathrm{MGR}(X)$ **or** $B \in \mathrm{MGR}(Y)$*).*
- $A \times B \in \mathrm{BP}(X \times Y) - \mathrm{MGR}(X \times Y)$
  if and only if
  ($A \in \mathrm{BP}(X) - \mathrm{MGR}(X)$ **and** $B \in \mathrm{BP}(Y) - \mathrm{MGR}(Y)$*).*

A subset $A \subseteq \prod_{n \in \mathbb{N}} X_n$ is called a *tail set* if for every $\alpha, \beta \in \prod_{n \in \mathbb{N}} X_n$ that differ on only finitely many coordinates, we have $\alpha \in A \Leftrightarrow \beta \in A$.

Then an important result called a $0-1$ law is proved: every set that is a tail set and has Baire Property, must either be meagre (small) or comeagre (big), see pages 7–8 of the lecture notes. This can be used to show that certain sets do not have Baire Property, because of their combinatorial structure.

Recall that a set is called *perfect* if it is closed and has no isolated points. In that case it must contain a copy of the Cantor set.

Exercise 5. *Let X be a perfect (no isolated points) Polish space. Construct a* Bernstein set *A in X such that neither A nor $X - A$ contain any non-empty perfect set.*

Sub-hint: order all the perfect sets in a sequence and proceed by transfinite induction.

Exercise 6. *Prove that if $A \subseteq X$ is a Bernstein set in a perfect Polish space X then A does not have Baire Property.*

An *ideal* on a set X is any non-empty family I of subsets of X that is closed under taking subsets and unions (an ideal is like a σ-ideal but without σ (countable unions)). We say that I is *proper* if $X \notin I$.

Notice that for each non-empty subset $C \in X$ the family $\{A \subseteq X \mid C \cap A = \varnothing\}$ is a maximal proper ideal. Such ideals are called *principal*. However, there are other ideals as well.

Exercise 7. *Show that there exists an ideal I on $\mathbb{N}$ that is maximal among those proper ideals that contain $\{A \subseteq \mathbb{N} \mid |A| < \infty\}$. We call such an ideal* $\mathrm{MAX} - \mathrm{FIN}$*.*

Notice that $\mathrm{MAX} - \mathrm{FIN}$ can be identified with the set of characteristic functions of its elements — a subset of 2^ω.

Exercise 8. *Prove that* $\mathrm{MAX} - \mathrm{FIN}$ *does not have Baire Property in* 2^ω.

You may use the fact that every ideal, viewed as a subset of 2^ω is a tail set.

The next part of the material is devoted to Mycielski–Kuratowski theorem and its variants:

Theorem 3.2. *If X is a non-empty perfect Polish spaces and A is comeagre in X^2 then there exists a copy of the Cantor set $C \subseteq X$ such that*

$$\left\{(x, y) \in C^2 \mid x \neq y\right\} \subseteq A.$$

As discussed in the lecture notes, the diagonal $\{(x, x) \mid x \in X\}$ is meagre in X^2 and therefore we need to exclude it from the left-hand side of the above formula.

Notice that if X is uncountable then X contains a perfect Polish space $X' \subseteq X$ as a subset. However, a comeagre subset of X might no longer be comeagre in X'. Thus, the assumption that X is perfect plays an important role here, as illustrated by the following exercise.

Exercise 9. *Give an example of a countable comeagre G subset of some (non-perfect) Polish space X.*

In such a case there cannot be any copy of the Cantor set inside G.

The proof of the Mycielski–Kuratowski theorem is based on an inductive construction of a Cantor scheme $(V_s)_{s \in 2^{<\omega}}$, whose limit set $\bigcup_{\alpha \in 2^\omega} \bigcap_{n \in \mathbb{N}} V_{\alpha \restriction_n}$ is C. During the construction, the following lemma is used that allows to adjust the sets appropriately: having fixed $(V_s)_{s \in 2^n}$ construct the sets $(V_s)_{s \in 2^{n+1}}$.

Exercise 10. *Assume that U is an open dense set in X^2 and let $\{G'_i \mid i \leqslant m\}$ be a finite collection of open non-empty sets in X. Then there exist open non-empty sets $G_i \subseteq G'_i$ for $i \leqslant m$ such that if $i \neq j$ then $G_i \times G_j \subseteq U$.*

Exercise 11. *Let $(\leqslant) \subseteq [0, 1] \times [0, 1]$ be a well-order of type ω_1 (it implies Continuum Hypothesis). Show that $\leqslant$ is not measurable w.r.t. the standard Lebesgue measure λ on $[0, 1]$.*

Show also that $\leqslant$ does not have Baire Property, because it contradicts Mycielski–Kuratowski theorem.

In fact the assumption on the shape of $\leqslant$ (ω_1) can be skipped and the set is still not measurable / does not have Baire property. However, it makes the arguments a bit simpler.

4 New homework

The above theorems, as stated, require our set to be comeagre in the product space X^n. We can weaken that by either allowing different spaces on different coordinates (i.e. $\prod_{i\leqslant n} X_i$) or considering A that is moderately big in the sense that $A \in \mathrm{BP}(X^n) - \mathrm{MGR}(X^n)$ — A needs to have Baire Property and not be small. These two variants are called Galvin theorems.

Exercise 12 (Galvin)**.** *Let $X_1, \ldots, X_n$ be non-empty perfect Polish spaces. If A is comeagre in $\prod_{i\leqslant n} X_i$ then there exist copies of the Cantor set $C_1, \ldots, C_n$ in $X_1, \ldots, X_n$ respectively such that $\prod_{i\leqslant n} C_i \subseteq A$.*

Let X be a non-empty perfect Polish space and $n \geqslant 1$. If $A \in \mathrm{BP}(X^n) - \mathrm{MGR}(X^n)$ (i.e. A has Baire Property but is not meagre) then there exist copies of the Cantor set $C_1, \ldots, C_n$ in X such that $\prod_{i\leqslant n} C_i \subseteq A$.

In the case of the above exercise, please be very careful and explicit in the argumentation — I will treat as a mistake statements like „analogously as in the lecture", „it is easy to check", etc... In other words, I'd like your solutions to be fully stand-alone and complete.

Please choose one exercise **or** another this time!

Exercise 13 ($\star$)**.** *Show that there exists a copy of the Cantor set $C \subseteq \mathbb{R}$ whose members are linearly independent over $\mathbb{Q}$: if $x_1, \ldots, x_n \in C$ are pairwise distinct and $\sum_{i\leqslant n} q_i \cdot x_i = 0$ for some rational numbers $q_1, \ldots, q_n \in \mathbb{Q}$ then all the numbers q_i equal 0.*

5 Hints

Hint to Exercise 4 Consider the first item. The right-to-left implication is obvious. Consider the left-to-right implication and assume that $A \notin \mathrm{MGR}(X)$. Then by Kuratowski–Ulam we know that

$$\mathrm{MGR}(Y) \ni \{y \in Y \mid (A \times B)^y \notin \mathrm{MGR}(X)\} = \\ = \{y \in Y \mid y \in B \wedge A \notin \mathrm{MGR}(X)\} = B.$$

For the second item, the right-to-left implication is obvious (please make sure that you know why $A \times B$ has Baire Property!). Assume that $A \times B \in \mathrm{BP}(X \times Y) - \mathrm{MGR}(X \times Y)$. Apply Kuratowski–Ulam to notice that if A does not have Baire property then B is meagre, in which case the first item implies that $A \times B$ is meagre — contradiction. Analogously, B must have Baire Property. Now, if any of them were meagre, again the first item would imply that $A \times B$ is meagre.

Hint to Exercise 5 As X has a countable basis, there is at most continuum many closed subsets of X. Consider an enumeration of all perfect subsets of X in a sequence $(F_\alpha)_{\alpha<\mathfrak{c}}$. Notice that each perfect subset of X has cardinality continuum itself. Proceed inductively for $\alpha < \mathfrak{c}$, constructing two sequences of points $x_\alpha, y_\alpha \in X$, such that $x_\alpha, y_\alpha \in F_\alpha$, and all the points $\{x_\alpha, y_\alpha \mid \alpha < \mathfrak{c}\}$ are pairwise distinct. At each stage, the cardinality of the set of points defined so far is $\alpha < \mathfrak{c}$ and the cardinality of F_α is $\mathfrak{c}$ so another two points $x_\alpha, y_\alpha \in F_\alpha$ exist. At the end of the construction, $A = \{x_\alpha \mid \alpha < \mathfrak{c}\}$ intersects each perfect set and the same holds for its complement (because of the points $y_\alpha \notin A$).

Hint to Exercise 6 Assume contrarily and by the symmetry assume that A is not meagre (otherwise take the complement of A). Then $A = U \triangle M$ for non-empty open U and meagre M contained in a meagre F_σ set F. Consider $U - F$ in the Polish space U. It is a G_δ set that is dense in U. If $U - F$ was countable then one could obtain U as a meagre F_σ: union of F with the singletons of members of $U - F$ (we use the fact that U is perfect here). Therefore, $U - F$ is an uncountable G_δ.

By the perfect set property for Polish spaces, it means that $U - F$ contains a perfect subset — this subset is contained in A, a contradiction.

Hint to Exercise 7 The union of a chain of ideals not containing $\mathbb{N}$ is an ideal not containing $\mathbb{N}$, so one can obtain a maximal ideal by Kuratowski–Zorn lemma.

Hint to Exercise 8 We first argue that $\mathrm{MAX} - \mathrm{FIN}$ is a tail set. Take two sequences (or sets) $\alpha, \beta \in 2^\omega$ that differ on finitely many positions. Assume by the symmetry that $\alpha \in \mathrm{MAX} - \mathrm{FIN}$. We need to show that $\beta \in \mathrm{MAX} - \mathrm{FIN}$ as well. However, there exists a finite set A_0 such that $\alpha \cup A_0 \supseteq \beta$ (treated as sets). As $A_0 \in \mathrm{MAX} - \mathrm{FIN}$, also $\alpha \cup A_0 \in \mathrm{MAX} - \mathrm{FIN}$ and therefore $\beta \in \mathrm{MAX} - \mathrm{FIN}$ as well.

Thus, if $\mathrm{MAX} - \mathrm{FIN}$ had Baire Property then it would either be meagre or comeagre. Observe that if $\alpha \in 2^\omega$ is any set and $\alpha^{\mathrm{c}} \in 2^\omega$ is the complement of α then (by maximality) $\mathrm{MAX} - \mathrm{FIN}$ contains exactly one of α, α^{c}.

Moreover, the function $\alpha \mapsto \alpha^{\mathrm{c}}$ is a homeomorphism of 2^ω. Thus, either both $\mathrm{MAX} - \mathrm{FIN}$ and its complement are meagre or both are comeagre. In both cases we have a contradiction.

Hint to Exercise 11 Notice that for each $y \in [0,1]$ the section $\leqslant^y$ is at most countable (because the order type of $\leqslant$ is ω_1). On the other hand, each section $\leqslant_x$ is co-countable. Therefore, all the sections $\leqslant^y$ have measure 0, while all the sections $\leqslant_x$ have measure 1. Contradiction to Fubini's theorem.

Hint to Exercise 9 Consider the space $2^{\leqslant\omega}$ of finite or infinite sequences. Let the topology be the one taken from the space 3^ω via the injection that maps $\alpha \in 2^\omega$ into itself, and $s \in 2^{<\omega}$ into $s^\frown 2^\frown 2^\frown 2 \cdots \in 3^\omega$. In this space, each finite sequence is an isolated point. Moreover, if $s_0 \prec s_1 \prec \ldots$ then this sequence is convergent to the limit $\bigcup_{n\in\mathbb{N}} s_n \in 2^\omega$. Since each $s \in 2^{<\omega}$ is an isolated point, the family $2^{<\omega}$ is an open subset of our space. Moreover, this set is dense, because of the above observation.

Hint to Exercise 10 This is an easy inductive argument: we can begin with $G_i = G_i'$ and then inductively make the sets smaller. For $j = 1, \ldots$ consider the current value of the set G_j and consider each of the previous sets G_i for $i < j$. Apply the following operation *shrink* to G_j and G_i:

given two non-empty open sets H and K in X and a dense open subset of X^2, there exists non-empty open subsets $\hat{H} \subseteq H$ and $\hat{K} \subseteq K$ such that $\hat{H} \times \hat{K} \subseteq U$, because U is dense so intersects $H \times K$ and in this intersection has some basic open set, which is of the form $V_1 \times V_2$ for some basic open sets V_1, V_2 of X. Take them as $\hat{H}$ and $\hat{K}$.