Tutorial from 16.04.2020

I'm really willing to answer your questions and comments.

I will grant additional points if someone indicates any "not-entirely-trivial" mistake in the notes!

Homework deadline: **24:00 on Wednesday 22.04**

1 Solutions of the homework problems

Exercise 1 ($\star$)**.** *Choose one of the two spaces* $[0,1]$ *or* 2^ω *and construct a dense* G_δ *set of* μ*-measure* 0*.*

The first option is to enumerate $\mathbb{Q} \cap [0,1] = \{q_0, q_1, \dots\}$ and put

$$U_n = \bigcup_{k \in \mathbb{N}} B(q_k, 1/n \cdot 2^{-k}).$$

By the fact that $\mathbb{Q} \cap [0,1] \subseteq U_n$ we know that U_n is dense (and clearly open). Therefore, $G \overset{\text{def}}{=} \bigcap_{n \in \mathbb{N}} U_n$ is a dense G_δ in $[0,1]$. However, the measure of U_n can be bounded from above by $2/n$ and therefore (by continuity of the measure) the measure of G is 0.

Another option is to go to 2^ω and define G as

$$G = \{\alpha \in 2^\omega \mid \alpha\restriction_n \in R \text{ for infinitely many } n\},$$

for properly defined $R \subseteq 2^{<\omega}$. Let $R = \bigcup_{n \in \mathbb{N}} R_n$ and let R_n contain sequences of the form $s\hat{\ }0^n$ where $|s| = 2^n$ (i.e. any sequence of length 2^n and then n consecutive zeros). Now estimate the probability that a random sequence $\alpha \in 2^\omega$ does not contain any prefix in R of length at least 2^n:

$$\mu\big\{\alpha \in 2^\omega \mid \forall k \geqslant 2^n.\ \alpha\restriction_k \notin R\big\} \leqslant 1 - \sum_{k \geqslant n} 2^{-k} = 1 - 2^{-n+1},$$

where the inequality follows from the fact that $\mu(R_k) = 2^{-k}$. Therefore,

$$\mu\big\{\alpha \in 2^\omega \mid \alpha\restriction_n \in R \text{ for only finitely many } n\big\} = 1.$$

And thus $\mu(G) = 0$.

Exercise 2 ($\star$)**.** *Take X non-empty at most countable. Prove that a set $A \subseteq X^\omega$ is comeagre if and only if the following condition holds. There exists a function $f\colon X^{<\omega} \to X^{<\omega} - \{<>\}$ such that for every sequence of sequences $(s_i) \in X^{<\omega}$ the following sequence belongs to A:*

$$s_0\hat{\ }f(r_0)\hat{\ }s_1\hat{\ }f(r_1)\hat{\ }s_2\hat{\ }f(r_2)\hat{\ }\cdots,$$

where r_i is defined inductively as $s_0\hat{\ }f(r_0)\hat{\ }s_1\hat{\ }f(r_1)\hat{\ }\ldots\hat{\ }f(r_{i-1})\hat{\ }s_i$.

This argument is based on the solution by Damian Gładkowski.

First assume that such a function f exists. Consider $R = \{s\hat{\ }f(s) \mid s \in X^{<\omega}\}$ and $G = \{\alpha \in X^\omega \mid \alpha{\restriction}_n \in R \text{ for infinitely many } n\}$. By some previous homework exercise G is a G_δ. We will show that G is dense and $G \subseteq A$.

Let $s \in X^{<\omega}$ be any sequence and define inductively $s_0 = s\hat{\ }f(s)$ and $s_{n+1} = s_n\hat{\ }f(s_n)$. Let $\alpha = \lim_{n\to\infty} s_n = \bigcup_{n\in\mathbb{N}} s_n$. Then clearly $s \prec \alpha$ and $\alpha \in G$, so $G \cap N_s \neq \varnothing$.

Now let $\alpha \in G$. Define inductively sequences (s_n) and (r_n) in such a way as in the statement (using the fact that infinitely many prefixes of α belong to R). Therefore, by the assumption $\alpha \in A$.

Now assume that A is a comeagre set and let $G \supseteq A$ be a dense G_δ. Let R be a set such that $G = \{\alpha \in X^{<\omega} \mid \alpha{\restriction}_n \in R \text{ for infinitely many } n\}$. Define f in such a way that for $s \in X^{<\omega}$ the value $f(s)$ is chosen in such a way that $s\hat{\ }f(s) \in R$ (it can be done by density of G). The function f defined this way satisfies the conditions because the constructed sequence must belong to G (infinitely many of its prefixes belong to R).

2 New material

First, the notion of a minimal cover is introduced: Consider a σ-algebra $\mathcal{C}$ and a σ-ideal $\mathcal{I} \subseteq \mathcal{C}$ (i.e. the σ-algebra already contains all the sets "modulo $\mathcal{I}$"). Then $(\mathcal{C}, \mathcal{I})$ has the *minimal cover property* if for **every** subset Y of our space there is a set $B \in \mathcal{C}$ such that:

- $Y \subseteq B$,
- if $C \in \mathcal{C}$ and $C \subseteq B - Y$ then $C \in \mathcal{I}$.

Notice that the set Y can be arbitrary and in particular we do not require that $B - Y \in \mathcal{I}$ because then Y would be forced to belong to $\mathcal{C}$.

Exercise 3. *Prove that the second condition from the definition can be equivalently restated as: if $C \in \mathcal{C}$ and $Y \subseteq C$ then $B - C \in \mathcal{I}$.*

Entail that if B is a minimal cover of Y then for every $F \in \mathcal{I}$ also $B \cup F$ is a minimal cover of Y.

Exercise 4. *Show that if B, B' are two minimal covers of Y then $B \triangle B' \in \mathcal{I}$.*

This means that the minimal cover is defined (among sets containing Y) up to $\mathcal{I}$-equivalence.

From that moment on we will focus on the case of $\mathcal{C}$ being $\mathcal{B}[\mathcal{I}]$ — completion of Borel sets by some σ-ideal $\mathcal{I}$. The σ-ideal $\mathcal{I}$ will be either $\mathcal{I}_\mu$ (for some σ-finite measure μ) or $\mathcal{I}_{\mathrm{MGR}}$.

The fact that $(\mathcal{B}, \mathcal{I}_\mu)$ and $(\mathcal{B}, \mathcal{I}_{\mathrm{MGR}})$ have minimal cover property is derived on page 2 of the notes from the fact that these σ-ideals are CCC.

It is later noted that when working with $\mathcal{I}_{\mathrm{MGR}}$ polishness of the topology is not required (Theorem on page 4 of the notes).

Assume that $\mathcal{I}$ is *generated* by Borel sets in the sense that every set in $\mathcal{I}$ is contained in a Borel set **in** $\mathcal{I}$.

Exercise 5. *Show that under the above assumptions, every set Y has a cover that is Borel itself.*

Exercise 6. *Show that the σ-ideal $\mathcal{I}_\mu$ of sets of measure 0 is generated by Borel sets.*

Exercise 7. *Show that the σ-ideal $\mathcal{I}_{\mathrm{MGR}}$ of meagre sets is generated by Borel sets.*

This means that in the above two cases we can always ask our cover to be Borel.

Our aim now is to prove the Marczewski theorem: if a pair $(\mathcal{C}, \mathcal{I})$ has minimal cover property then $\mathcal{C}$ is closed under the Souslin operation $\mathcal{A}$.

For that, two lemmata (Lemma 1 and 2 from the notes) are needed.

Exercise 8. *Let $A = \mathcal{A}(B_s)$. Then*

$$B_\varnothing - A \subseteq \bigcup_{s \in \mathbb{N}^{<\omega}} \big(B_s - \bigcup_{n \in \mathbb{N}} B_{s\hat{}n}\big).$$

Exercise 9. *Give an example of a regular (i.e. monotone) scheme where the inclusion above is strict.*

Exercise 10. *If for every $s \in \mathbb{N}^{<\omega}$ we have $B_s \subseteq \bigcup_{n\in\mathbb{N}} B_{s\hat{}n}$ then $B_\varnothing = \mathcal{A}(B_s)$.*

Exercise 11. *For every Souslin scheme (A_s) there exists the* faithful *scheme (Z_t) defined as $Z_t = \mathcal{A}\big(A_{t\hat{}s}\big)_{s\in\mathbb{N}^{<\omega}}$. This scheme satisfies:*

1. $Z_\varnothing = \mathcal{A}(A_s)$;
2. $Z_t = \bigcup_{n\in\mathbb{N}} Z_{t\hat{}n}$;
3. $Z_t \subseteq A_t$.

The only problem with the scheme (Z_t) is that we have no reason to assume that $Z_t \in \mathcal{C}$ — this scheme is already obtained using the Souslin operation $\mathcal{A}$!

Having proved those lemmas, we can provide a proof of Marczewski's theorem using an interplay of three schemes:

1. (A_s) is the initial scheme that we begin with;
2. (Z_s) is the *faithful* scheme as in Lemma 2 (see Exercise 11 above), with Z_t being the set of points that $\mathcal{A}(A_s)$ produces for branches $\alpha \succ t$;
3. (B_s) is a scheme of approximations: B_s is the intersection of some minimal cover of Z_s with A_s.

Comment about B_s: we know that the minimal cover is defined up-to $\mathcal{I}$ and a priori might be bigger than A_s even though $Z_s \subseteq A_s$. However, intersecting with A_s doesn't cost anything because A_s itself belongs to $\mathcal{C}$. Moreover, B_s defined that way is still a minimal cover of Z_s because $Z_s \subseteq A_s$. In general, we can always intersect our minimal cover with any set from $\mathcal{C}$ that we have at hand, as long as we still contain Z_s — the smaller the cover the better.

The construction ensures that we have $Z_s \subseteq B_s \subseteq A_s$ for each $s \in \mathbb{N}^{<\omega}$. This implies that $\mathcal{A}(Z_s) = \mathcal{A}(B_s) = \mathcal{A}(A_s)$, where the external equality is clear and implies the equality with $\mathcal{A}(B_s)$.

Now we invoke Exercise 8. If one knew that for each $s \in \mathbb{N}^{<\omega}$ we had $B_s - \bigcup_{n\in\mathbb{N}} B_{s\hat{}n} \in \mathcal{I}$ then $B_\varnothing - A \in \mathcal{I}$ and A is measurable itself.

Exercise 12. *Take $s \in \mathbb{N}^{<\omega}$ and prove that $B_s - \bigcup_{n\in\mathbb{N}} B_{s\hat{}n} \in \mathcal{I}$.*

3 New homework

I strongly believe that the first exercise is easier than the second one. But again, you are **allowed** to solve both, because both are entertaining :)

Exercise 13 ($\star$)**.** *Let $X \subseteq 2^\omega$ be a set such that for every $\alpha \in 2^\omega$ and every $n \in \mathbb{N}$ we have*

$$\alpha \in X \Leftrightarrow \alpha' \notin X,$$

where α' is the same as α except that $\alpha'(n) = 1 - \alpha(n)$. In other words, for every two sequences $\alpha, \alpha' \in 2^\omega$ that differ on **exactly** *one position, X contains exactly one of them. Prove that such a set exists using the axiom of choice. Prove that every set X satisfying the above condition does not have Baire Property.*

The set constructed above can be seen as a variant of Vitali set known from analysis.

Exercise 14 ($\star$)**.** *Let $\sigma(\mathbf{\Sigma}^1_1)$ be the σ-algebra generated by analytic sets $\mathbf{\Sigma}^1_1$. Let $\mathcal{A}(\mathbf{\Pi}^1_1)$ be the family of sets obtained via the Souslin operation applied to coanalytic sets. Prove that $\sigma(\mathbf{\Sigma}^1_1) \subsetneq \mathcal{A}(\mathbf{\Pi}^1_1)$ (two tasks: prove the inclusion and prove that it is strict).*

Going further, one can define a hierarchy of sets (called $\mathcal{C}$-sets, possibly because of **C**omprehension. . .) with: $\mathcal{C}_0 = \mathcal{B}$ (Borel sets), and $\mathcal{C}_{n+1} = \mathcal{A}\big(\mathcal{C}_n^{\mathrm{c}}\big)$ — the Souslin operation applied to complements of sets from $\mathcal{C}_n$. This way $\mathcal{C}_1 = \mathbf{\Sigma}^1_1$, etc. The above exercise can be extended to show that the σ-algebras $\sigma(\mathcal{C}_n)$ form a strictly increasing sequence of σ-algebras. Moreover, all the sets in that hierarchy are measurable and have Baire Property!

One can relatively easily see that all the sets $\mathcal{C}_n$ defined above are contained in $\mathbf{\Delta}^1_2 = \mathbf{\Sigma}^1_2 \cap \mathbf{\Pi}^1_2$ — they can be obtained as projections of coanalytic sets and coprojections of analytic sets. From that perspective, Marczewski's theorem is (almost) at the frontier of our knowledge about measurability and Baire Property because there are universes of set theory where certain $\mathbf{\Delta}^1_2$ sets are **not** measurable and do **not** have Baire Property!

4 Hints

Hint to Exercise 3 Should be easy, because $\mathcal{I}$ is closed under finite unions.

Hint to Exercise 4 Again finite unions.

Hint to Exercise 5 Take a minimal cover B of Y. Since $B \in \mathcal{B}[\mathcal{I}]$, we know that $B = A \triangle F$ for $A \in \mathcal{B}$ and $F \in \mathcal{I}$. Let $F' \in \mathcal{I} \cap \mathcal{B}$ be a Borel set that contains F. Then $A \cup F' \in \mathcal{B}$ and $B \subseteq A \cup F'$. We claim that $A \cup F'$ is the required cover. Take $H \subseteq (A \cup F') - Y$ and observe that H can be split into a subset of F' (that belongs to $\mathcal{I}$ because F' does), a subset of $A \cap F$ (again the same), and a subset of $B - Y$ that belongs to $\mathcal{I}$ because B was a minimal cover.

Hint to Exercise 6 Clear, because we know that one can approximate every set by a G_δ from above with the same measure. So every set of measure 0 is contained in a G_δ of measure 0.

Hint to Exercise 7 This time every meagre set is contained in a meagre F_σ.

Hint to Exercise 8 Assume contrarily and take $x \in B_\varnothing - A$ that does not belong to any of the sets $B_s - \bigcup_{n\in\mathbb{N}} B_{s\hat{\ }n}$. Construct inductively $s_0 \prec s_1 \prec \dots$ such that $x \in B_{s_i}$. Begin with $s_0 = \varnothing$ and continue using the fact that $x \notin B_{s_i} - \bigcup_{n\in\mathbb{N}} B_{s_i\hat{\ }n}$. Thus, we can put $\alpha = \lim_{i\to\infty} s_i = \bigcup_{i\in\mathbb{N}} s_i$ and observe that $x \in \bigcap_{n\in\mathbb{N}} B_{\alpha\restriction_n}$ what means that $x \in A$.

Hint to Exercise 9 Take for instance $B_\varnothing = B_{0\hat{\ }s} = B_1 = X$ for each $s \in \mathbb{N}^{<\omega}$ and $B_t = \varnothing$ for all the remaining t. Then $\mathcal{A}(B_s) = X = A$ and $B_\varnothing - A = \varnothing$. However, the considered union contains for instance

$$B_1 - \bigcup_{n\in\mathbb{N}} B_{1\hat{\ }n} = X - \varnothing = X.$$

Hint to Exercise 10 Apply directly Exercise 8.

Hint to Exercise 11 Each of the statements is obvious from the choice of Z_t :)

Hint to Exercise 12 We know that B_s is a minimal cover of Z_s, the same for $B_{s\hat{}n}$ and $Z_{s\hat{}n}$. Denote the difference $D = B_s - \bigcup_{n\in\mathbb{N}} B_{s\hat{}n}$. We need to show that $D \in \mathcal{I}$. For each n, since $D \cap B_{s\hat{}n} = \emptyset$ for each n we know that also $D \cap Z_{s\hat{}n} = \emptyset$ (because $Z_t \subseteq B_t$). But $Z_s = \bigcup_{n\in\mathbb{N}} Z_{s\hat{}n}$ what means that $D \cap Z_s = \emptyset$. Thus, $D \subseteq B_s - Z_s$ and D is $\mathcal{C}$ measurable. Thus, by the fact that B_s is a minimal cover, we know that $D \in \mathcal{I}$.