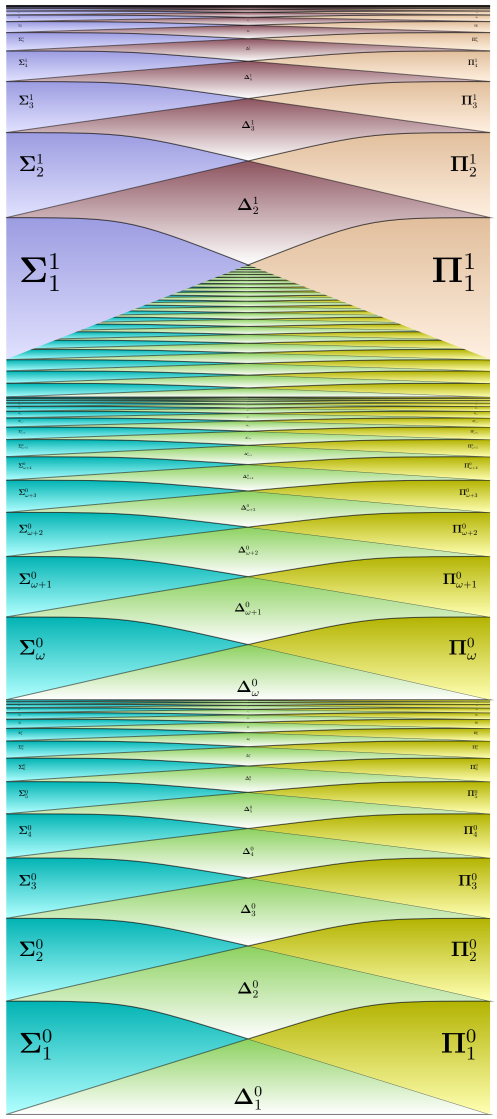


The above figure represents the Borel and projective hierarchies. At the moment we know: Borel sets $\left(\boldsymbol{\Sigma}^0_\eta, \boldsymbol{\Delta}^0_\eta, \boldsymbol{\Pi}^0_\eta\right)_{\eta<\omega_1}$, analytic sets $\boldsymbol{\Sigma}^1_1$, and coanalytic sets $\boldsymbol{\Pi}^1_1$. We additionally know that $\boldsymbol{\Delta}^1_1 = \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1$ coincides with Borel sets $\mathcal{B}$.

Similarly as $\boldsymbol{\Sigma}^1_1$ sets are projections of Borel sets, one can define $\boldsymbol{\Sigma}^1_2$ as projections of coanalytic sets, $\boldsymbol{\Pi}^1_2$ as complements of $\boldsymbol{\Sigma}^1_2$, etc… Analogously, one puts $\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n$. The hierarchy of families $\left(\boldsymbol{\Sigma}^1_n, \boldsymbol{\Delta}^1_n, \boldsymbol{\Pi}^1_n\right)_{n\in\mathbb{N}}$ is called the *projective hierarchy*.

## Tutorial from 02.04.2020

I'm really willing to answer your questions and comments.

I will grant additional points if someone indicates any "not-entirely-trivial" mistake in the notes!

Homework deadline: **24:00 on Wednesday 08.04**

# 1 Solutions of the homework problems

**Exercise 1.** *Consider a "binary Souslin scheme" $(A_s)_{s\in 2^{<\omega}}$. You can think of it as a general Souslin scheme $A_s$ such that $A_s = \varnothing$ whenever $s \notin 2^{<\omega}$. Prove that if all the sets $A_s$ are Borel then also $\mathcal{A}_2(A_s)$ (this subscript 2 indicates that the scheme is binary) is Borel.*

W.l.o.g. assume that the scheme $(A_s)_{s\in 2^{<\omega}}$ is *regular*, i.e. if $s \preceq r$ then $A_s \supseteq A_r$. We claim that $\mathcal{A}_2(A_s) = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} A_s$. Then, $\mathcal{A}_2(A_s)$ is Borel as a countable intersection of finite unions of Borel sets $A_s$. The inclusion $\mathcal{A}_2(A_s) \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} A_s$ is obvious. Take $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} A_s$ and let

$$T = \{s \in 2^{<\omega} \mid x \in A_s\}.$$

It is clear that $T$ is prefix closed. Since $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} A_s$, $T$ is infinite. Therefore, König's Lemma implies that $T$ has an infinite branch $\alpha \in 2^\omega$. Therefore, $x \in \bigcap_{n\in\mathbb{N}} A_{\alpha\restriction_n} \subseteq \mathcal{A}_2(A_s)$.

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**Exercise 2** ($\star$)**.** *Consider another two variants of Souslin operation: given a scheme $(A_s)_{s\in\mathbb{N}^{<\omega}} \subseteq X$, put:*

$$\mathcal{A}^{(\infty)}(A_s) = \bigcup_{\alpha\in\mathbb{N}^\omega} \{x \in X \mid x \in A_{\alpha\restriction_n} \text{ for infinitely many } n\},$$
$$\mathcal{A}_2^{(\infty)}(A_s) = \bigcup_{\alpha\in 2^\omega} \{x \in X \mid x \in A_{\alpha\restriction_n} \text{ for infinitely many } n\}.$$

*W.l.o.g. the argument for the operation $\mathcal{A}_2^{(\infty)}$ is a binary scheme $(A_s)_{s\in 2^{<\omega}}$.*

*Consider a family of sets $\Gamma \subseteq \mathsf{P}(X)$ that is closed under finite unions and finite intersections and contains $\varnothing$ and $X$. Prove that*

$$\mathcal{A}\Gamma = \mathcal{A}^{(\infty)}\Gamma = \mathcal{A}_2^{(\infty)}\Gamma, \tag{1.1}$$

*i.e. exactly the same family of sets can be obtained via $\mathcal{A}$, $\mathcal{A}^{(\infty)}$, and $\mathcal{A}_2^{(\infty)}$ applied to all the possible schemes from $\Gamma$.*

We will show three containments. They are a bit boring but the important message is that the Souslin operation is about witnesses $\alpha$ and what we do here is encode witnesses of one type as witnesses of other type.

$\mathcal{A}\Gamma \subseteq \mathcal{A}^{(\infty)}\Gamma$ This is clear if we make our scheme regular: given $(A_s)_{s\in\mathbb{N}^{<\omega}}$ define $A'_s = \bigcap_{t\leq s} A_t$ using finite intersections in $\Gamma$. Then for each $\alpha \in \mathbb{N}^\omega$ we have $\bigcap_{n\in\mathbb{N}} A'_{\alpha\restriction_n} = \{x \in X \mid x \in A'_{\alpha\restriction_n} \text{ for infinitely many } n\}$ and therefore $\mathcal{A}(A_s) = \mathcal{A}(A'_s) = \mathcal{A}^{(\infty)}(A'_s)$.

$\mathcal{A}^{(\infty)}\Gamma \subseteq \mathcal{A}_2^{(\infty)}\Gamma$ Consider a scheme $(A_s)_{s\in\mathbb{N}^{<\omega}}$. We will encode natural numbers $n \in \mathbb{N}$ as binary sequences $0^n1 \in \{0,1\}^{<\omega} = 2^{<\omega}$, where $0^n$ is the sequence consisting of $n$ zeros. Let

$$\phi\colon \mathbb{N}^{\leqslant\omega} \ni \langle n_0, n_1, \ldots\rangle \mapsto 0^{n_0}10^{n_1}1\ldots \in 2^{\leqslant\omega},$$

where by $X^{\leqslant\omega}$ we mean $X^{<\omega} \cup X^\omega$ — in fact these are two functions, one mapping finite sequences to finite sequences and the other mapping infinite sequences to infinite sequences. Notice that $\phi$ is a bijection between $\mathbb{N}^\omega$ and $G = \{\alpha \in 2^\omega \mid \alpha \text{ has infinitely many ones}\}$. Also, $\phi$ is 1-1 on $\mathbb{N}^{<\omega}$. Now, for each $r \in 2^{<\omega} \cap \mathrm{rg}(\phi)$ put $B_r = A_{\phi^{-1}(r)}$ and for $r \in 2^{<\omega} - \mathrm{rg}(\phi)$ put $B_r = \varnothing$.

We claim that $\mathcal{A}_2^{(\infty)}(B_r) = \mathcal{A}^{(\infty)}(A_s)$. For the $(\supseteq)$ inclusion take $x \in X$ and $\alpha \in \mathbb{N}^\omega$ such that for infinitely many $n \in I$ we have $x \in A_{\alpha\restriction_n}$. Let $\beta = \phi(\alpha)$. Notice that for each $n \in \mathbb{N}$ we have $\phi(\alpha\restriction_n) \prec \beta$. In particular, for each $n \in I$ we have $x \in B_{\phi(\alpha\restriction_n)}$ and the sequences $\phi(\alpha\restriction_n)$ are distinct for distinct $n$. Therefore, there is infinitely many $m$ such that $x \in B_{\beta\restriction_m}$ and thus $x \in \mathcal{A}_2^{(\infty)}(B_r)$.

Now consider the $(\subseteq)$ inclusion: take $\beta \in 2^\omega$ such that for infinitely many $m \in I$ we have $x \in B_{\beta\restriction_m}$. Notice that $\beta \in G$ as otherwise $\beta$ would have only finitely many prefixes in $\mathrm{rg}(\phi)$. In fact, for each $m \in I$ we must have $\beta\restriction_m \in \mathrm{rg}(\phi)$.

Let $\alpha = \phi^{-1}(\beta)$. We claim that $x \in A_{\alpha\restriction_n}$ for infinitely many $n$. But again, for each $m \in I$ we have $x \in A_{\phi^{-1}(\beta\restriction_m)}$, again $\phi^{-1}(\beta\restriction_m) \prec \alpha$, and for distinct $m$ the sequences $\phi^{-1}(\beta\restriction_m)$ are distinct. So $x \in A_{\alpha\restriction_n}$ for infinitely many $n$.

$\mathcal{A}_2^{(\infty)}\Gamma \subseteq \mathcal{A}\Gamma$ We will again encode witnesses for $\mathcal{A}_2^{(\infty)}$ as witnesses for $\mathcal{A}$. Let $(B_r)_{r\in 2^{<\omega}}$ be a binary Souslin scheme. Fix a bijection $\psi\colon \mathbb{N} \to 2^{<\omega} - \{<>\}$

that enumerates all non-empty finite binary sequences. Put

$$A_{<n_0,\dots,n_k>} = B_{\psi(n_0)\hat{}\psi(n_1)\hat{}\dots\hat{}\psi(n_k)}.$$

We claim that $\mathcal{A}_2^{(\infty)}(B_r) = \mathcal{A}(A_s)$. If $x \in \mathcal{A}_2^{(\infty)}(B_r)$ then there exists $\beta = s_0\hat{}s_1\hat{}\dots \in 2^\omega$ with all the sequences $s_k \in 2^{<\omega}$ non-empty such that for each $k \in \mathbb{N}$ we have $x \in B_{s_0\hat{}s_1\hat{}\dots\hat{}s_{k-1}}$. Let $\alpha =< \psi^{-1}(s_0), \psi^{-1}(s_1), \dots >\in \mathbb{N}^\omega$. For each $k \in \mathbb{N}$ we have $A_{\alpha\restriction_k} = B_{s_0\hat{}s_1\hat{}\dots\hat{}s_{k-1}}$. Therefore, for each $k$ we know that $x \in A_{\alpha\restriction_k}$ and thus $x \in \mathcal{A}(A_s)$.

Now assume that $x \in \mathcal{A}(A_s)$, as witnessed by $\alpha =< n_0, n_1, \dots >\in \mathbb{N}^\omega$. Let $s_k = \psi(n_k)$ for $k = 0, \dots$ and put $\beta = s_0\hat{}s_1\hat{}\dots$ As all the sequences $s_k$ are non-empty, we know that $\beta \in 2^\omega$. Notice that for each $k \in \mathbb{N}$ we have $B_{s_0\hat{}s_1\hat{}\dots\hat{}s_{k-1}} = A_{<n_0,\dots,n_{k-1}}$, what means that $x \in B_{s_0\hat{}s_1\hat{}\dots\hat{}s_{k-1}}$. Therefore, the set $\{s_0\hat{}s_1\hat{}\dots\hat{}s_{k-1} \mid k \in \mathbb{N}\}$ is an infinite set of prefixes of $\beta$ where $x$ belongs to the respective set $B_r$. Therefore, $x \in \mathcal{A}_2^{(\infty)}(B_r)$.

# 2 New material

## 2.1 Abstract $\sigma$-ideals

A *$\sigma$-ideal* is a non-empty family $\mathcal{I} \subseteq \mathsf{P}(X)$ that is closed under subsets (if $A \subseteq A' \in \mathcal{I}$ then $A \in \mathcal{I}$) and countable unions (if $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{I}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}$). In particular, we always have $\varnothing \in \mathcal{I}$ and also $\mathcal{I}$ is closed under arbitrary intersections of non-empty families of sets.

A typical example is $\mathcal{I}_{\aleph_0} = \{A \subseteq X \mid |A| \leqslant \aleph_0\}$ — the $\sigma$-ideal of at most countable sets. A typical non-example is the ideal of finite sets $\mathcal{I}_{\text{fin}} = \{A \subseteq X \mid |A| < \infty\}$ — it is closed under finite unions but may not be closed under countable unions.

Let $\mathcal{A}$ be a $\sigma$-algebra (closed under countable operations and complement) and $\mathcal{I}$ be a $\sigma$-ideal. Define the family

$$\mathcal{A}[\mathcal{I}] = \{A \triangle N \mid A \in \mathcal{A}, N \in \mathcal{I}\},$$

where $B \triangle C = (B - C) \cup (C - B)$.

**Exercise 3.** *Let $A \in \mathcal{A}[\mathcal{I}]$ and $A \triangle B \in \mathcal{I}$. Show that then $B \in \mathcal{A}[\mathcal{I}]$. In other words, $\mathcal{A}[\mathcal{I}]$ contains whole equivalence classes of the relation*

$$A \sim_{\mathcal{I}} B \iff A \triangle B \in \mathcal{I}.$$

*(make sure that $\sim_{\mathcal{I}}$ is in fact an equivalence relation!)*

**Exercise 4.** *Check that under the above assumptions, $\mathcal{A}[\mathcal{I}]$ is a $\sigma$-algebra.*

**Exercise 5.** *Entail that $\mathcal{A}[\mathcal{I}]$ is the smallest $\sigma$-algebra containing $\mathcal{A} \cup \mathcal{I}$.*

## 2.2 Borel measures

A Borel measure $\mu$ is a function from $\mathcal{B}(X)$ to $[0, +\infty]$. Such a function assigns a measure only to Borel sets (trivium). However, various interesting sets (e.g. analytic) are not Borel. Also, it is often the case that $B \in \mathcal{B}(X)$, $\mu(B) = 0$, and $A \subseteq B$ but $A \notin \mathcal{B}(X)$; so the value $\mu(A)$ is undefined (although we expect it to be 0)!

This leads to the idea: consider $\mathcal{I}_\mu = \{A \subseteq X \mid \exists B \in \mathcal{B}(X).\ A \subseteq B \wedge \mu(B) = 0\}$ — the $\sigma$-ideal of sets of $\mu$-measure zero.

**Exercise 6.** *Check that $\mathcal{I}_\mu$ is a $\sigma$-ideal.*

We call a set $A \subseteq X$ *$\mu$-measurable* if $A = B \triangle N$ for $B \in \mathcal{B}(X)$ and $N \in \mathcal{I}_\mu$. This means that the $\mu$-measurable sets is the $\sigma$-algebra $\mathcal{B}[\mathcal{I}_\mu]$. One can extend $\mu$ into $\bar{\mu}$ on $\mathcal{B}[\mathcal{I}_\mu]$ by putting

$$\bar{\mu}(B \triangle N) = \mu(B).$$

**Exercise 7.** *Prove that the above formula does not depend on the representation of the argument.*

Notice that while the family of Borel sets is fixed for a given space, the family of $\mu$-measurable sets depends on the actual values of the measure: if $\mu([7, 9]) = 0$ then **every** subset of the interval $[7, 9]$ is $\mu$-measurable. There is a more robust notion of *universally measurable set*: it is $\mu$-measurable **for every** Borel measure $\mu$. Thus, the aim of the lecture is to ultimately prove that every analytic set is universally measurable.

**Exercise 8.** *Check that $\bar{\mu}$ is also a $\sigma$-additive measure:*

1. $\bar{(\mu)}(\varnothing) = 0$,
2. $\bar{\mu}(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \bar{\mu}(A_n)$ *for each collection of pairwise disjoint $\mu$-measurable sets $A_n$.*

The lecture notes provide a notion of a *ccc $\sigma$-ideal in* $\mathcal{B}[\mathcal{I}]$ that has no uncountable family of pairwise disjoint sets in $\mathcal{B}[\mathcal{I}] - \mathcal{I}$.

**Exercise 9.** *Prove that there is no uncountable family of pairwise disjoint sets in $\mathcal{B}[\mathcal{I}] - \mathcal{I}$ if and only if there is no such family in $\mathcal{B} - \mathcal{I}$.*

Hint: you can enumerate your hypothetical uncountable family by the first uncountable ordinal $\omega_1$, which means that you work with a sequence of sets $(B_\eta)_{\eta<\omega_1}$.

**Exercise 10.** *Assume the conditions of Exercise 9. Prove that there is no uncountable family of almost-pairwise disjoint sets in $\mathcal{B}[\mathcal{I}] - \mathcal{I}$ (two sets $B$ and $B'$ are* almost disjoint *if $B \cap B' \in \mathcal{I}$).*

Then, the lecture notes provide a construction of compact sets that approximate a given measurable set from below up to $\varepsilon > 0$. During this construction, one fixes for each $n \in \mathbb{N}$ an at most countable family of closed balls $B_i^{(n)}$ that cover our space $X$ and $\operatorname{diam}(B_i^{(n)}) \leqslant 2^{-n}$. Then one can take sufficiently large $k_n$ such that

$$\mu\big(X - \bigcup_{i\leqslant k_n} B_i^{(n)}\big) < \varepsilon \cdot 2^{-n-1}.$$

Then one puts $K = \bigcap_{n\in\mathbb{N}} \bigcup_{i\leqslant k_n} B_i^{(n)}$ and observes that $\mu(X - K) \leqslant \varepsilon$. It remains to see that $K$ is compact. It is clearly closed (as an intersection of finite (!) unions of closed sets).

**Exercise 11.** *Show that $K$ is totally bounded, i.e. for every $\delta > 0$ there exists a finite set $F \subseteq K$ such that $K \subseteq \bigcup_{x\in F} B(x, \delta)$.*

Now, by taking a union of $K_\varepsilon$ for countably many $\varepsilon \to 0$ (e.g. $\varepsilon = 1/n$) one obtains an $F_\sigma$ that is contained in a given set and has the same measure as it.

## 2.3 Meagre sets

Another important $\sigma$-ideal is that of meagre (also written meager or *first category*) sets. The lecture notes define a *meagre* set as a set that is a countable union of nowhere dense sets ($A$ is nowhere dense if $\operatorname{int}(\overline{A}) = \varnothing$, i.e. its closure does not contain any non-empty open set).

One can equivalently define a set $A \subseteq X$ to be *meagre* if there exists a countable family of closed sets $(F_n)_{n\in\mathbb{N}}$ such that for every $n$ we have $\operatorname{int}(F_n) = \varnothing$ (i.e. no non-empty open set is contained in $F_n$) and $A \subseteq \bigcup_{n\in\mathbb{N}} F_n$.

**Exercise 12.** *Prove that a set is meagre in the above meaning if and only if it meagre in the sense of the lecture notes.*

A set is *comeagre* if its complement is meagre. They are considered BIG.

**Exercise 13.** *Let $X$ be a topological space that might not be Polish. Prove that the following three conditions are equivalent:*

1. *every non-empty open set is non-meagre;*
2. *every comeagre set is dense;*
3. *the intersection of a countable family of dense open sets is dense.*

Recall the Baire category theorem in terms of $G_\delta$s for Polish spaces.

**Theorem 2.1.** *If $X$ is a Polish space and for each $n \in \mathbb{N}$ a set $G_n \subseteq X$ is a dense $G_\delta$ then $\bigcap_{n\in\mathbb{N}} G_n$ is also a dense $G_\delta$.*

A space satisfying the conditions of Exercise 13 is called a *Baire space*. Baire category theorem states that each Polish space is a special case of a Baire space. We again assume all our spaces to be Polish, as generally during this lecture.

**Exercise 14.** *Prove that a set $A$ is comeagre if and only if it contains some dense $G_\delta$.*

**Exercise 15.** *Prove that meagre sets constitute a $\sigma$-ideal, denoted here $\mathcal{I}_{\mathrm{MGR}}$.*

Comeagre sets form a $\sigma$-filter — a concept dual to $\sigma$-ideal.

This leads to two definitions: first, we can apply the standard construction of $\mathcal{B}[\mathcal{I}_{\mathrm{MGR}}]$ obtaining the smallest $\sigma$-algebra that contains $\mathcal{B}(X)$ and $\mathcal{I}_{\mathrm{MGR}}$. Second, one can consider the family of sets $\mathrm{BP}(X)$ that have *Baire property*:

$$\{U \triangle N \mid U \in \mathbf{\Sigma}^0_1, N \in \mathcal{I}_{\mathrm{MGR}}\},$$

i.e. sets that are open modulo $\mathcal{I}_{\mathrm{MGR}}$. Clearly, $\mathrm{BP}(X) \subseteq \mathcal{B}[\mathcal{I}_{\mathrm{MGR}}]$. Proposition on pages 8/9 of lecture notes prove the opposite containment. In particular, given $(U_n \triangle N_n)_{n\in\mathbb{N}}$ with $U_n$s open, one needs to be able to construct an open set $U$ such that

$$\left(\bigcap_{n\in\mathbb{N}} (U_n \triangle N_n)\right) \triangle U \in \mathcal{I}_{\mathrm{MGR}}.$$

Then again a variant of approximation theorem for $\mathrm{BP}(X)$ comes, but now the roles of $G_\delta$s and $F_\sigma$s are swapped comparing to the measure case.

The proposition at the end of page 10 is again a variant of Exercise 7, saying that each set is either in $\mathcal{I}_{\mathrm{MGR}}$ or of the form $U \triangle \mathcal{I}_{\mathrm{MGR}}$ for **non-empty** open set $U$; but not both.

Finally, the lecture notes again show the lack of uncountable anti-chains w.r.t. $\mathcal{I}_{\mathrm{MGR}}$, as in Exercise 9.

### 2.4 Making measurable functions continuous

Please note that I'm **not** summarising all the contents of the lecture notes: there was considerably more (i.e. making measurable functions continuous on large sets).

The first variant of the result (by Souslin) works for a Borel measure $\mu$, see page 6 of lecture notes.

**Exercise 16.** *Check that $g$ defined as in the proof is in fact continuous.*

A very similar construction is given also at pages 12–13 for Baire-measurable functions.

## 3 New homework

Probably the first exercise is a bit easier but I'm not sure.

**If someone wants, you can solve both of them this week :)**

Consider two measurable spaces. The first is $\langle [0,1], \lambda \rangle$ — the interval $[0,1] \subseteq \mathbb{R}$ with the standard Lebesgue measure $\lambda$. The second is $\langle 2^\omega, \mu \rangle$, where the probability Borel measure $\mu$ corresponds to randomly and independently choosing bits: $\mu(N_s) = 2^{-|s|}$ for each $s \in 2^{<\omega}$.

**Exercise 17 ($\star$).** *Choose one of the two spaces defined above and construct a dense $G_\delta$ set of $\mu$-measure 0.*

The set constructed above is BIG w.r.t. $\mathcal{I}_{\mathrm{MGR}}$ but small w.r.t. $\mathcal{I}_\mu$. Its complement has the opposite properties.

**Exercise 18** ($\star$)**.** *Take $X$ non-empty at most countable. Prove that a set $A \subseteq X^\omega$ is comeagre if and only if the following condition holds. There exists a function $f\colon X^{<\omega} \to X^{<\omega} - \{<>\}$ such that for every sequence of sequences $(s_i) \in X^{<\omega}$ the following sequence belongs to $A$:*

$$s_0 \,\hat{}\, f(r_0) \,\hat{}\, s_1 \,\hat{}\, f(r_1) \,\hat{}\, s_2 \,\hat{}\, f(r_2) \,\hat{}\, \cdots ,$$

*where $r_i$ is defined inductively as $s_0 \,\hat{}\, f(r_0) \,\hat{}\, s_1 \,\hat{}\, f(r_1) \,\hat{}\, \ldots \,\hat{}\, f(r_{i-1}) \,\hat{}\, s_i$.*

Anti-hint: the function $f$ above is in fact a *strategy*...

# 4 Hints

**Hint to Exercise 3** Simple calculation. In fact it works also for ideals (that are closed only under finite unions).

**Hint to Exercise 4** First, $\mathcal{A}[\mathcal{I}]$ is closed under complement, because $(A\triangle N)^{\mathrm{c}} = A^{\mathrm{c}}\triangle N$. Therefore, it is enough to show that $\mathcal{A}[\mathcal{I}]$ is closed under countable unions. Take $(A_n\triangle N_n)_{n\in\mathbb{N}}$. Then check that

$$\left(\bigcup_{n\in\mathbb{N}}(A_n\triangle N_n)\right) \triangle \left(\bigcup_{n\in\mathbb{N}} A_n\right) \subseteq \bigcup_{n\in\mathbb{N}} N_n \in \mathcal{I} \tag{4.1}$$

**Hint to Exercise 5** We have shown above that $\mathcal{A}[\mathcal{I}]$ is a $\sigma$-algebra. Clearly, each of its elements $(A\triangle N)$ must exist to any $\sigma$-algebra that contains $\mathcal{A} \cup \mathcal{I}$. Therefore, its the smallest one.

**Hint to Exercise 6** By $\sigma$-additivity of measure, if $A_n \subseteq B_n$ and $\mu(B_n) = 0$ then $B = \bigcup_{n\in\mathbb{N}} B_n$ is also Borel and $\mu(B) = 0$ and $\bigcup_{n\in\mathbb{N}} A_n \subseteq B$.

**Hint to Exercise 7** If $B\triangle N = B'\triangle N'$ then $B\triangle B' \in \mathcal{I}$ (write an actual formula for this difference in terms of $N$ and $N'$!). This implies that $\mu(B) = \mu(B')$ (why?).

**Hint to Exercise 8** One only needs to check countable additivity. But it is clear from (4.1) (is it?).

**Hint to Exercise 9** First, one implication is clear. Using the hint after the exercise, assume contrarily and take a family of pairwise disjoint sets $(B_\eta \triangle N_\eta)_{\eta<\omega_1}$ in $\mathcal{B}[\mathcal{I}] - \mathcal{I}$ (i.e. each $B_\eta$ is Borel, each $N_\eta$ is in $\mathcal{I}$, and their symmetric difference is not in $\mathcal{I}$). To reach a contradiction we need to construct an analogous family $(B'_\eta)_{\eta<\omega_1}$ of sets in $\mathcal{B} - \mathcal{I}$.

Define inductively, $B'_\eta = B_\eta - \bigcup_{\tau<\eta} B'_\tau$. Since each $\eta$ itself is a countable ordinal, we know that all $B'_\eta$ are Borel. They are clearly pairwise disjoint by the definition. It remains to see that no $B'_\eta \in \mathcal{I}$. Assume contrarily, that $B'_\eta \in \mathcal{I}$ for some $\eta < \omega_1$. Since each $B'_\tau \subseteq B_\tau$, we have

$$\mathcal{I} \ni B'_\eta \supseteq B_\eta - \bigcup_{\tau<\eta} B_\tau. \tag{4.2}$$

But as $(B_\tau \triangle N_\tau)$ are pairwise disjoin, $B_\eta \cap B_\tau \in \mathcal{I}$ for each $\eta < \tau$. Therefore, $B_\eta - \bigcup_{\tau<\eta} B_\tau = B_\eta - N$ for some $N \in \mathcal{I}$. But then $B_\eta - N \in \mathcal{I}$ by (4.2). This means that also $B_\eta \in \mathcal{I}$ and therefore also $B_\eta \triangle N_\eta \in \mathcal{I}$. A contradiction.

Notice that the above construction depends heavily on the fact that each $\eta$ is a countable ordinal, so at each stage of the construction we need to deal with countably many "previous" sets. This comes from the fact that we have chosen our family to be ordered by $\omega_1$ — the whole family is uncountable, but each set has countably many predecessors. It is a frequent trick in set theory.

**Hint to Exercise 10** The argument is essentially the same as above: we can inductively normalise the sets $B_\eta$ by removing from them $\bigcup_{\tau<\eta} B_\eta \cap B_\tau$ — this is a member of $\mathcal{I}$, because $\eta$ itself is a countable ordinal.

**Hint to Exercise 11** Take $\delta > 0$ and $n$ sufficiently big so that $2^{-n} < \delta/2$. Then $K \subseteq \bigcup_{i \leqslant k_n} B_i^{(n)}$. For each of the balls $B_i^{(n)}$ for $i \leqslant k_n$, either $K \cap B_i^{(n)} = \varnothing$ and we can skip this ball, or it is non-empty and we can take to $F$ any of its points. Then $|F| \leqslant k_n$.

**Hint to Exercise 12** If $A \subseteq \bigcup_{n\in\mathbb{N}} F_n$ as in our definition, then each $A \cap F_n$ is nowhere dense itself, because $\overline{A \cap F_n} \subseteq F_n$ and $\mathrm{int}(F_n) = \varnothing$.

Now, if $A$ is a countable union of nowhere dense sets $A_n$ then the closures $\overline{A_n}$ have empty interiors and can be used as $F_n$s.

**Hint to Exercise 13** Assume (1) and take $A$ comeagre. Assume that $A$ is not dense: there exists open $U$ such that $A \cap U = \varnothing$. Then $U \subseteq X - A$ is meagre. Contradiction.

Assume (2) and take a countable family $(U_n)_{n\in\mathbb{N}}$ of dense open sets. Then for each $n$ the set $X - U_n$ is nowhere dense, because $\mathrm{int}(\overline{X - U_n}) = \mathrm{int}(X - U_n) = \varnothing$ (the last equality follows from the fact that $U_n$ is dense). This means that $\bigcup_{n\in\mathbb{N}}(X - U_n)$ is meagre and therefore $\bigcap_{n\in\mathbb{N}} U_n = X - \bigcup_{n\in\mathbb{N}}(X - U_n)$ is comeagre and therefore dense by (2).

Now assume (3) and for the sake of contradiction assume that $U = \bigcup_{n\in\mathbb{N}} A_n$ is a nono-empty open set, with each $A_n$ nowhere dense. Take $U_n = X - \overline{A_n}$. Since $\mathrm{int}(\overline{A_n}) = \varnothing$, we know that $U_n$ is dense and open. Therefore, $\bigcap_{n\in\mathbb{N}} U_n$ is dense in $X$ by (3). This means that it intersects $U$, which is a contradiction ($\overline{A_n} \supseteq A_n$).

**Hint to Exercise 14** By dualising the definition with $F_n$s — the complement of a closed set $F_n$ with empty interior is a dense open set. Their intersection is a dense $G_\delta$ due to Baire theorem.

**Hint to Exercise 15** Take a countable family of meagre sets $A_n$. The complement of each of them contains a dense $G_\delta$. The intersection of all these $G_\delta$s is again a dense $G_\delta$. Moreover, $\bigcup_{n\in\mathbb{N}} A_n$ is disjoint from that intersection.

**Hint to Exercise 16** It is enough to check that $g^{-1}(V_n)$ is open in $B$ for each basic open set $V_n$. However, we know that $f^{-1}(V_n) \subseteq U_n$ and $U_n - f^{-1}(V_n) \subseteq B_n$. By the first inclusion we know that $g^{-1}(V_n) \subseteq U_n \cap B$ (as $B$ is the domain of $g$). Take $x \in U_n \cap B$ and observe that $x \notin B_n$ and therefore $f(x) \in V_n$, so $x \in g^{-1}(V_n)$. This means that $g^{-1}(V_n) = U_n \cap B$ (as in the lecture notes) which is an open set in the induced topology on $B \subseteq X$.