Tutorials from 26.03.2020

In case of any questions, please contact me by e-mail!
Homework deadline: **24:00 on Wednesday 01.04**

1 Solutions of the homework problems

Exercise 1. *Prove that if* Γ *is a boldface pointclass that has a universal set*[1] $U \subseteq X \times X$ *then* $\Gamma \neq \Gamma^c$.
You can use the Cantor's diagonal argument from page 13 of the lecture notes.

Take $U \subseteq X \times X$ universal. Then $X \times X - U \in \Gamma^c$. Let $\varphi(x) = (x, x)$ and $V = \varphi^{-1}(X \times X - U)$. We know that $V \in \Gamma^c$ because both Γ and Γ^c are boldface pointclasses. We will show that $V \notin \Gamma(X)$. Assume contrarily. Then $V = U_y$ for some $y \in X$. But then:

$$(y, y) \in U \Leftrightarrow \in U_y \Leftrightarrow y \in V \Leftrightarrow (y, y) \notin U,$$

where the first two equivalence comes from the definition of U_y, the second from the assumption that $V = U_y$, and the third from the definition of V. Contradiction.

Exercise 2 ($\star$)**.** *Prove that the following set is universal for* $\mathbf{\Sigma}^1_1$*:*

$$\left\{(T, \alpha) \in \mathrm{Tr}_{\mathbb{N}\times\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \mid \exists \beta \in \mathbb{N}^{\mathbb{N}}.\ (\alpha, \beta) \in [T]\right\}.$$

You may follow the idea (left as an exercise) from pages 11–12 of the lecture notes.

It follows easily from the universality of

$$\left\{(T, \alpha) \in \mathrm{Tr}_X \times X^{\omega} \mid \forall n \in \mathbb{N}.\ \alpha\restriction_n \in T\right\}.$$

2 New material

2.1 Souslin's separation theorem

Theorem 2.1. *If* A *and* B *are two disjoint analytic sets* $\mathbf{\Sigma}^1_1$ *then there exists a Borel set separating them.*

[1] For technical reasons we assume that $X = Y$ here.

The proof given during the lecture is not constructive: in the middle of page 2 it is said "suppose that A, B are **not** Borel separable". One can provide a bit more constructive proof, as follows.

Again take $A = f(\mathbb{N}^\omega)$ and $B = g(\mathbb{N}^\omega)$ for two continuous functions $f, g\colon \mathbb{N}^\omega \to X$ and put $A_s = f(N_s)$ and $B_s = g(N_s)$. Define

$$T = \{s \in \mathbb{N}^{<\omega} \mid A_s \text{ and } B_s \text{ cannot be separated by an open set}\}.$$

Clearly T is a tree (it is prefix closed).

Exercise 3. *Show that $[T] = \varnothing$.*

(The idea is the same as in the lecture notes)

Exercise 4. *If $T \in \mathrm{Tr}_{\mathbb{N}}$ is a non-empty tree that is* well founded *(has no infinite branch) then T has a* leaf*: an element $s \in T$ such that $\forall n \in \mathbb{N}.\ s\hat{}n \notin T$.*

Now we can inductively "shrink" the tree: for each leaf s of T one can apply Lemma from page 1 of lecture notes to find a Borel separator of A_s and B_s and remove s from T. Continue like that (transfinite induction) removing consecutive leaves of T. In the limit we must get $T = \varnothing$ what means that we've constructed a separator of $A_{<>}$ and $B_{<>}$. This shows how transfinite levels of Borel hierarchy can be involved in a construction of the separator.

Exercise 5. *Prove that there are two disjoint analytic sets A and B that cannot be separated by any set from a finite level of Borel hierarchy (i.e. a set from $\mathbf{\Sigma}^0_n$ for some n).*

As a consequence of the separation result, we get the following important fact that was shown during the lecture.

Exercise 6. *If both A and $X - A$ are analytic (in other words A is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$) then A is Borel.*

2.2 Souslin schemes

A Souslin scheme is a family of sets $(A_s)_{s\in\mathbb{N}^{<\omega}} \subseteq X$ indexed by finite sequences of naturals. You should think of A_s as somehow *definable* (e.g. Borel). Then

$$\mathcal{A}\big((A_s)_{s\in\mathbb{N}^{<\omega}}\big) = \bigcup_{\alpha\in\mathbb{N}^\omega} \bigcap_{n\in\mathbb{N}} A_{\alpha\restriction_n} = \big\{x \in X \mid \exists \alpha \in \mathbb{N}^\omega.\ \forall n \in \mathbb{N}.\ x \in A_{\alpha\restriction_n}\big\}.$$

This clearly implies that $\mathcal{AB}(X) \subseteq \mathbf{\Sigma}^1_1(X)$: if we apply the Souslin operation to Borel sets we get only analytic sets. Lecture notes prove that $\mathbf{\Sigma}^1_1(X) \subseteq \mathcal{A}F(X)$ — every analytic set can be obtained via Souslin operation applied to closed sets.

Souslin operation $\mathcal{A}$ turns out to be very well behaved. Take any set X and any family of sets $\Gamma \subseteq \mathsf{P}(X)$. Assume that Γ contains $\varnothing$ and X.

Exercise 7. *Prove that* $\Gamma \subseteq \mathcal{A}\Gamma$.

The following exercise is a bit easier for unions but the intersections should also be fine (see a hint after the statement).

Exercise 8. *Prove that* $\mathcal{A}\Gamma$ *is closed under countable unions and intersections.*

Try to "shuffle" the schemes (A^k_s) in such a way that each branch witnesses all of them at once. You might use the fact that:

$$\bigcap_{k\in\mathbb{N}} \bigcup_{\alpha\in\mathbb{N}^\omega} \bigcap_{n\in\mathbb{N}} A^k_{\alpha\restriction_n} = \bigcup_{\theta\in(\mathbb{N}^\omega)^{\mathbb{N}}} \bigcap_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} A^k_{\theta(k)\restriction_n}. \tag{2.1}$$

Exercise 9. *Show that* $\mathcal{AA}\Gamma = \mathcal{A}\Gamma$.

Therefore, we know that

$$\mathbf{\Sigma}^1_1(X) = \mathcal{A}F(X) = \mathcal{AB}(X) = \mathcal{AAB}(X) = \mathcal{A}\mathbf{\Sigma}^1_1(X)$$

We have seen that every analytic set $B \in \mathbf{\Sigma}^1_1(X)$ is a projection of a closed set F:

$$B = \{x \in X \mid \exists \alpha \in \mathbb{N}^\omega.\ (x, \alpha) \in F\}.$$

Therefore, $x \in B$ iff $\exists \alpha \in \mathbb{N}^\omega.\ (x, \alpha) \in F$ iff

$$\exists n_0 \in \mathbb{N}.\ \exists n_1 \in \mathbb{N}.\ \cdots\ \big(x, (n_0, n_1, \ldots)\big) \in F$$

(I'm aware that this infinite sequence of quantifiers does not make sense) Now take $B \in \mathcal{A}F(X)$ that is obtained as $B = \mathcal{A}(F_s)$ for a Souslin scheme (F_s) of closed sets. W.l.o.g assume that $F_{<>} = X$. Then $x \in B$ iff

$$\exists n_0 \colon x \in F_{(n_0)}.\ \exists n_1 \colon x \in F_{(n_0,n_1)}.\ \cdots\ \underbrace{\top}_{truth}.$$

This suggests that the Souslin operation is in general weaker than projection. It is actually the case, and in particular $\mathcal{A}\mathbf{\Pi}^1_1$ is strictly contained in $\mathbf{\Sigma}^1_2$, we might come back to that later.

Exercise 10. *Consider a scheme* (B_s) *such that* $B_s \cap B_t = \emptyset$ *for each* n *and* $s, t \in \mathbb{N}^n$, $s \neq t$ *and whenever* $t \preceq s$ *then* $B_t \supseteq B_s$. *Prove that*

$$\mathcal{A}(B_s) = \bigcap_n \bigcup_{s \in \mathbb{N}^n} B_s.$$

Exercise 11. *Prove that if a scheme* (B_s) *is Borel and satisfies the above conditions then* $\mathcal{A}(B_s)$ *is also Borel.*

The above observation is at the core of the proof of the following Lusin–Souslin Theorem from page 10 of lecture notes (here stated in terms of projection).

Theorem 2.2. *Let* $B \subseteq X \times Y$ *be Borel and let* $A = \pi_X(B)$. *Assume that for every* $x \in X$ *the section* B_x *has cardinality at most one (this property says that* B *is* uniformised *or it is a graph of a partial function). Then* A *is also Borel.*

During the proof, one needs to make a given topology zero dimensional: take X as a Polish space. Then, the topology τ on X can be extended into another Polish topology τ' such that τ' is zero dimensional (has a basis of clopens).

The idea of the proof is to take a countable basis $\{U_n \mid n \in \mathbb{N}\}$ of τ and put $F_n = X - U_n$ — this set is closed, in particular Borel. By results of previous lectures, one can extend τ into τ_n that is Polish and F_n is open in τ_n. Then, the topology τ' generated by the union of all these topologies is also Polish.

Exercise 12. *Show that* τ' *is in fact zero dimensional.*

3 New homework

Exercise 13. *Consider a "binary Souslin scheme"* $(A_s)_{s \in 2^{<\omega}}$. *You can think of it as a general Souslin scheme* A_s *such that* $A_s = \emptyset$ *whenever* $s \notin 2^{<\omega}$. *Prove that if all the sets* A_s *are Borel then also* $\mathcal{A}_2(A_s)$ *(this subscript* 2 *indicates that the scheme is binary) is Borel.*

Exercise 14 ($\star$). *Consider another two variants of Souslin operation: given a scheme* $(A_s)_{s\in\mathbb{N}^{<\omega}} \subseteq X$, *put:*

$$\mathcal{A}^{(\infty)}(A_s) = \bigcup_{\alpha\in\mathbb{N}^\omega} \{x \in X \mid x \in A_{\alpha\restriction_n} \text{ for infinitely many } n\},$$
$$\mathcal{A}_2^{(\infty)}(A_s) = \bigcup_{\alpha\in 2^\omega} \{x \in X \mid x \in A_{\alpha\restriction_n} \text{ for infinitely many } n\}.$$

W.l.o.g. the argument for the operation $\mathcal{A}_2^{(\infty)}$ *is a binary scheme* $(A_s)_{s\in 2^{<\omega}}$.

Consider a family of sets $\Gamma \subseteq \mathsf{P}(X)$ *that is closed under finite unions and finite intersections and contains* $\varnothing$ *and* X. *Prove that*

$$\mathcal{A}\Gamma = \mathcal{A}^{(\infty)}\Gamma = \mathcal{A}_2^{(\infty)}\Gamma, \tag{3.1}$$

i.e. exactly the same family of sets can be obtained via $\mathcal{A}$, $\mathcal{A}^{(\infty)}$, *and* $\mathcal{A}_2^{(\infty)}$ *applied to all the possible schemes*[2] *from* Γ.

In the last exercise I will grant points also for partial solutions: non-trivial inclusions in (3.1).

4 Hints

Hint to Exercise 3 Assume contrarily that $\alpha \in [T]$. Then (similarly as in the lecture notes), $f(\alpha) \neq g(\alpha)$. But then $f(\alpha)$ and $g(\alpha)$ can be separated by open sets and by continuity of f and g it holds for some finite $s \preceq \alpha$. Thus, $s \notin T$.

Hint to Exercise 4 Go inductively down the tree. You either reach a leaf, or construct an infinite branch.

Hint to Exercise 5 Take any set B that is in $\mathbf{\Sigma}^0_{\omega+7}$ but not in $\mathbf{\Sigma}^0_{\omega+6}$. Consider $A = X - B$. Then the only separator of A and B is the set A itself and it is not $\mathbf{\Sigma}^0_n$ for any $n \in \mathbb{N}$.

Hint to Exercise 6 Apply the separation theorem to those sets. Again, the only separator is A.

[2] Of course it may happen that $\mathcal{A}^{(\infty)}(A_s) \neq \mathcal{A}(A_s)$ for a fixed scheme (A_s).

Hint to Exercise 7 It is enough to take $A \in \Gamma$ and put a scheme (A_s) constantly equal A.

Hint to Exercise 8 For the union, take a sequence of schemes $(A_s^k)_{s\in\mathbb{N}^{<\omega},k\in\mathbb{N}}$. Define a new scheme with $B_{<>} = X$ and $B_{\hat{k}s} = A_s^k$ — the scheme (B_s) seen as a tree has root labelled X and each subtree of the root under k is a copy of the scheme $(A_s^k)_{s\in\mathbb{N}^{<\omega}}$. Then

$$\mathcal{A}(B_s) = \bigcup_{\alpha\in\mathbb{N}^\omega}\bigcap_{n\in\mathbb{N}} B_{\alpha\restriction_n} = \bigcup_{k\in\mathbb{N}}\bigcup_{\beta\in\mathbb{N}^\omega} X \cap \bigcap_{n\in\mathbb{N}} B_{\hat{k}\beta\restriction_n} = \bigcup_{k\in\mathbb{N}} \mathcal{A}(A_s^k),$$

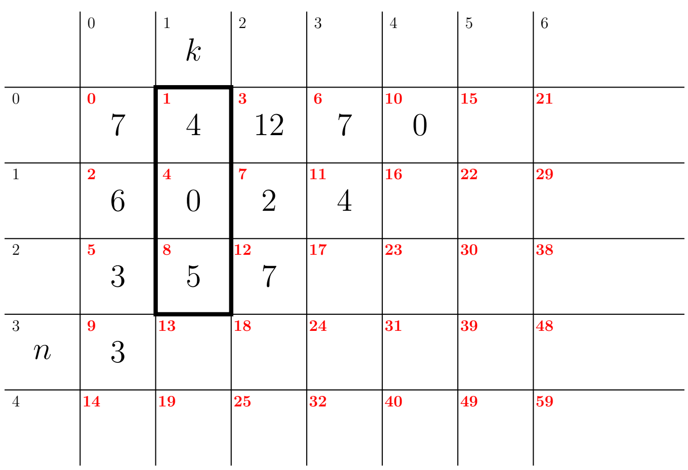
where α is split into the form $\hat{n}\beta$.

Now for the intersection, take again a sequence of schemes $(A_s^k)_{s\in\mathbb{N}^{<\omega},k\in\mathbb{N}}$.

Enumerate $\mathbb{N}^2$ by natural numbers using $\iota\colon \mathbb{N}^2 \to \mathbb{N}$. Assume that this enumeration is monotone on both coordinates (it goes in the zyg-zag fashion). This means, that given sequence $s \in \mathbb{N}^i$ with $s = \langle s(0), \dots, s(i-1)\rangle$ we can take $(k,n) = \iota^{-1}(i)$ (we call k the *last active column of* s) and then extract from s its coordinates numbered $\iota(k,0)$, $\iota(k,1)$, …, $\iota(k,n-1)$. Let $\widehat{s} = s(\iota(k,0))\dots s(\iota(k,n-1)) \in \mathbb{N}^n$ be the sequence of values of s on those coordinates.

The following picture depicts an example of this situation for $s \in \mathbb{N}^{13}$ as below.

$$s = \langle\, 7\,,\ 4\,,\ 6\,,\ 12,\ 0\,,\ 3\,,\ 7\,,\ 2\,,\ 5\,,\ 3\,,\ 0\,,\ 4\,,\ 7\,\rangle$$



The small numbers in upper-left corner of each cell indicate the coordinates of that cell: the left-right axis indicates k; the top-down axis indicates

n; and the red boldface numbers inside the grid are $\iota(k, n)$. Our i equals 13 because $s \in \mathbb{N}^{13}$. Since $\iota^{-1}(13) = (1, 3)$, we have $k = 1$ and $n = 3$. Therefore, the last active column of s is $k = 1$. The coordinates that we want to extract from s are $\iota(1, 0) = 1$, $\iota(1, 1) = 4$, and $\iota(1, 2) = 8$. Therefore, $\widehat{s} = \langle 4, 0, 5 \rangle \in \mathbb{N}^3$.

Consider a new scheme (B_s) that is defined for $s \in \mathbb{N}^{<\omega}$ in the following way. Let k be the last active column of s and let $B_s = A^k_{\widehat{s}}$.

Claim 4.1. *We have*

$$\mathcal{A}(B_s) = \bigcap_{k \in \mathbb{N}} \mathcal{A}(A^k_s).$$

To prove that we will define a bijection between $(\mathbb{N}^\omega)^{\mathbb{N}}$ and $\mathbb{N}^\omega$. We will denote it ϕ. Take $(\alpha_k)_{k \in \mathbb{N}}$ and for $i \in \mathbb{N}$ with $\iota^{-1}(i) = (k_i, n_i)$ define $\phi\big((\alpha_k)_{k \in \mathbb{N}}\big)(i) = \alpha_{k_i}(n_i)$. As ι is a bijection on the set of coordinates, ϕ is a bijection on the level of sequences. Thus, it is enough to prove the following claim for $x \in X$, $(\alpha_k)_{k \in \mathbb{N}}$, and $\beta = \phi\big((\alpha_k)_{k \in \mathbb{N}}\big)$:

$$x \in \bigcap_{i \in \mathbb{N}} B_{\beta \upharpoonright_i} \iff \forall k \in \mathbb{N}.\ x \in \bigcap_{n \in \mathbb{N}} A_{\alpha_k \upharpoonright_n} \tag{4.1}$$

Take any $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $i = \iota(k, n)$ and let $s = \beta\upharpoonright_i$. Then k is the last active column of s and $B_s = A^k_{\widehat{s}}$. However, by the choice of ϕ, we know that $\widehat{s} = \alpha_k\upharpoonright_n$. Therefore, $x \in A_{\alpha_k \upharpoonright_n} \Leftrightarrow x \in B_s$. [this paragraph can be read twice, to get the two implications in (4.1)]

Hint to Exercise 9 Again, we can encode in $\alpha \in \mathbb{N}^\omega$ another sequence $\beta \in \mathbb{N}^\omega$ together with a sequence of witnesses $\alpha_n \in \mathbb{N}^\omega$ for $n \in \mathbb{N}$.

This proof is based on the argument in Kechris, Proposition 25.6.

Clearly it is enough to show that $\mathcal{A}\mathcal{A}\Gamma \subseteq \mathcal{A}\Gamma$. Let $A = \mathcal{A}(s \mapsto P_s)$ with $P_s \in \mathcal{A}\Gamma$, i.e. $P_s = \mathcal{A}(t \mapsto Q_{s,t})$ (to avoid confusion, by $\mathcal{A}(r \mapsto A_r)$ I denote the Souslin operation on the parameter r). It is easy to check that

$$x \in A \iff \exists \alpha \in \mathbb{N}^\omega.\ \exists \beta \in (\mathbb{N}^\omega)^\omega.\ \forall m.\ \forall n.\ x \in Q_{\alpha\upharpoonright_m, z(m)\upharpoonright_n}.$$

Fix a bijection $\iota \colon \mathbb{N}^2 \to \mathbb{N}$ such that $m \leqslant \iota(m, n)$ and $p < n \Rightarrow \iota(m, p) < \iota(m, n)$ (the zig-zag function from page 6 of tutorial_3_26.pdf is good). Let ι_0^{-1} and ι_1^{-1} be the respective coordinates of the reverse function, i.e. for every $k \in \mathbb{N}$ we have $\iota\big(\iota_0^{-1}(k), \iota_1^{-1}(k)\big) = k$.

Our aim is to encode witnesses $(\alpha, \beta) \in \mathbb{N}^\omega \times (\mathbb{N}^\omega)^\omega$ as single sequences in $\mathbb{N}^\omega$ using the above function shuffling the coordinates. We will encode $(\alpha, \beta) \in \mathbb{N}^\omega \times (\mathbb{N}^\omega)^\omega$ by $w \in \mathbb{N}^\omega$ defined as

$$w(k) \stackrel{\text{def}}{=} \iota\big(\alpha(k), \beta(\iota_0^{-1}(k))(\iota_1^{-1}(k))\big).$$

Notice a tiny difference with the previous approach: we not only mix the coordinates using ι but also mix the actual values: a single number (coordinate) in w codes a coordinate of α together with a coordinate of one of the sequences β. This gives the desired bijection.

Note that if we know $w\restriction_{\iota(m,n)}$ then we can determine $\alpha\restriction_m$ (because each coordinate of α goes into w via ι) and also $\beta(m)\restriction_n$ (because the function ι is sufficiently monotone). This gives rise to a pair of functions $\varphi, \psi\colon \mathbb{N}^{<\omega} \to \mathbb{N}^{<\omega}$ such that if w encodes (α, β) in the above sense and $s = w\restriction_{\iota(m,n)}$ then $\varphi(s) = \alpha\restriction_m$ and $\psi(s) = \beta(m)\restriction_n$ (notice that the length of s determines the values of m and n).

Put $R_s = Q_{\varphi(s),\psi(s)}$ and notice that

$$x \in A \Leftrightarrow x \in \mathcal{A}(s \mapsto R_s).$$

Hint to Exercise 10 The inclusion $\subseteq$ is clear. Take $x \in \bigcap_n \bigcup_{s\in\mathbb{N}^n} B_s$. By the assumption of disjointness, there is a unique $s_1 \in \mathbb{N}^1$ such that $x \in B_{s_1}$. Similarly, there is a unique $s_2 \in \mathbb{N}^2$ such that $x \in B_{s_2}$, moreover $s_1 \prec s_2$. This allows us to inductively construct a branch $\alpha \in \mathbb{N}^\omega$ such that for every n we have $x \in B_{\alpha\restriction_n}$. Therefore, $x \in \mathcal{A}(B_s)$.

Hint to Exercise 11 That is obvious, because the right-hand side of the formula in Exercise 6 is obtained from the sets B_s via countable operations.

Hint to Exercise 12 First we need to consider finite intersections of sets in the basis of $\tau \cup \{F_n\}$. Those are either basic sets of τ, or of the form $F_n \cap U$ for some U basic in τ. Thus, this is a basis of the topology τ_n. Now take finite intersections of those sets. They are either basic sets of τ, or of the form:

$$U_k \cap F_{i_1} \cap \ldots \cap F_{i_n},$$

for some $k, n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbb{N}$. The complement of such a set equals

$$F_k \cup U_{i_1} \cup \ldots \cup U_{i_n},$$

which is an open set in τ'. Therefore, the basis of τ' consists of clopens.