This file provides materials of the tutorial that was expected on 19th March. Please study this file carefully before the next lecture (26th March).

In case of any questions, please contact me by e-mail. I'm willing to explain as much as needed either by mail or some more online tool (shared whiteboard etc).

The deadline for the new homework is this Wednesday 24:00 (but the homework should be easy) :)

1 Solutions of the homework problems

Exercise 1. *Let* Ord *be the subset of* $2^{\mathbb{N}\times\mathbb{N}}$ *that contains* $o \in 2^{\mathbb{N}\times\mathbb{N}}$ *(treated as a subset of* $\mathbb{N}\times\mathbb{N}$ *if and only if* o *is a linear order. Show that* Ord *is a closed subset of* $2^{\mathbb{N}\times\mathbb{N}}$.

Consider $\mathrm{IO} \subseteq \mathrm{Ord}$ *that contains* o *if and only if* o *is not well-founded (it contains a subset that has no minimal element). Show that* IO *is an analytic set* $\mathbf{\Sigma}^1_1$.

The fact that Ord is closed follows directly from the definition of a linear order: this definition is of the form $\forall_{x,y,z}.\ \varphi(x,y,z)$, where $\varphi(x,y,z)$ is a clopen property of o. More formally, let $F_{x,y,z}$ contain $o \in 2^{\mathbb{N}\times\mathbb{N}}$ if and only if the following conditions hold (we write $o(x,y)$ to denote $o(x,y)=1$):

- if $o(x,y)$ and $o(y,z)$ then $o(x,z)$,
- $o(x,x)$,
- if[1] $x \neq y$ then either $\neg o(x,y)$ or $\neg o(y,x)$.

Notice that for each $x,y,z \in \mathbb{N}$ the set $F_{x,y,z}$ is a clopen in $2^{\mathbb{N}\times\mathbb{N}}$, because the fact whether $o \in F_{x,y,z}$ depends only on the five values (or coordinates) $o(x,y)$, $o(y,z)$, $o(x,z)$, $o(x,x)$, and $o(y,x)$.

Now we can observe that

$$\mathrm{Ord} = \bigcap_{x,y,z\in\mathbb{N}} F_{x,y,z}$$

is a countable intersection of closed sets and it is itself closed.

[1] One might write here: if $o(x,y)$ and $o(y,x)$ then $x=y$. In this notation the condition $x=y$ does not depend on o, so in the case of each set $F_{x,y,z}$ separately it is either constantly true or constantly false.

Now we need to show that IO is $\mathbf{\Sigma}^1_1$. Let $Y = \mathbb{N}^{\mathbb{N}}$ and consider the following set

$$F = \Big\{(o,\alpha) \in \mathrm{Ord} \times Y \mid \forall i < j.\ \alpha(i) \neq \alpha(j) \wedge o\big(\alpha(j), \alpha(i)\big)\Big\}.$$

Again it can be written as

$$F = \bigcap_{i<j\in\mathbb{N}} F_{i,j}, \tag{1.1}$$

where

$$F_{i,j} = \Big\{(o,\alpha) \in \mathrm{Ord} \times Y \mid \alpha(i) \neq \alpha(j) \wedge o\big(\alpha(j), \alpha(i)\big)\Big\}.$$

[[The trick of changing logical quantifiers into set-theoretic operations is frequent in Descriptive Set Theory.]]

Claim 1.1. *Each of the sets $F_{i,j}$ is again a clopen.*

Proof. Take any pair $(o,\alpha) \in \mathrm{Ord} \times Y$. Fix four coordinates: i, j, $\alpha(i)$, and $\alpha(j)$. Consider any other pair $(o',\alpha') \in \mathrm{Ord}\times Y$ such that $\alpha(i) = \alpha'(i)$, $\alpha(j) = \alpha'(j)$, and $o(\alpha(j),\alpha(i)) = o'(\alpha(j),\alpha(i))$. Then $(o,\alpha) \in F_{i,j} \Leftrightarrow (o',\alpha') \in F_{i,j}$.

This means that every member of $F_{i,j}$ belongs to that set together with some neighbourhood; and the same for the complement of $F_{i,j}$. Therefore, $F_{i,j}$ is clopen. ■

This, together with the formula (1.1) implies that F is closed. Now observe that $(o,\alpha) \in F$ if and only if the consecutive numbers in α form an infinite strictly descending chain in o. Thus,

$$\big(o \in \mathrm{IO}\big) \iff \big(\exists \alpha \in \mathbb{N}^{\mathbb{N}}.\ (o,\alpha) \in F\big) \iff \big(o \in \pi_{\mathrm{Ord}}(F)\big).$$

Exercise 2 ($\star$)**.** *Show that the set of trees over* 2 *with an infinite branch (i.e. ill-founded binary trees in* Tr_2*) is a Borel subset of* Tr_2*.*

This set is in fact closed, as the projection of the closed set

$$F = \{(T,\alpha) \in \mathrm{Tr}_2 \times 2^\omega \mid \forall n \in \mathbb{N}.\ \alpha{\restriction}_n \in T.\},$$

see Exercise 3 below.

Another way of solving this problem is the following: by König's Lemma[2] we know that $T \in \mathrm{Tr}_2$ has an infinite branch if and only if

$$\forall n \in \mathbb{N}.\ \exists s \in 2^n.\ s \in T.$$

Therefore, again we have

$$\mathrm{IF}_2 = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} \{T \in \mathrm{Tr}_2 \mid s \in T\},$$

which is a Borel representation of IF_2 (the final set $\{T \in \mathrm{Tr}_2 \mid s \in T\}$, parametrised by $s \in 2^n$, is clopen).

If you look carefully at that formula, the last union $\bigcup_{s\in 2^n}$ is in fact finite! Therefore, the set $\bigcup_{s\in 2^n}\{T \in \mathrm{Tr}_2 \mid s \in T\}$ is also clopen and the whole set is closed.

2 New material

We begin by an exercise related to the last-week problems.

Exercise 3. *Show that if $F \subseteq X \times Y$ is closed and Y is compact then the projection $\pi_X(F)$ is closed in X. Notice that we do not assume X (and thereafter $X \times Y$) to be compact!*

A typical example is the set

$$F = \big\{(x, y) \in \mathbb{R} \times [-\pi/2, +\pi/2] \mid x = \tan(|y|)\big\}$$

with $\pi_{\mathbb{R}}(F) = [0, \infty)$ that is closed but not compact.

2.1 Borel hierarchy

During the lecture the notion of *Borel sets* (the smallest σ-algebra containing all open sets) was defined. We will now provide a more constructive way of representing them.

First, let $\mathbf{\Sigma}^0_1(X)$ be the family of open sets in a Polish space X. Now, we put $\mathbf{\Pi}^0_\eta(X)$ as the family of complements of the sets in $\mathbf{\Sigma}^0_\eta(X)$, and $\mathbf{\Sigma}^0_{\eta+1}(X)$ as the family of all sets that are countable unions of sets in $\mathbf{\Pi}^0_\eta(X)$.

[2] König's Lemma $\equiv$ compactness; you may want to track down the counterpart of König-like argument in a solution of Exercise 3.

Now we go transfinite: for any η that is a countable limit ordinal, we put $\mathbf{\Sigma}^0_\eta(X)$ to be the family of sets that can be obtained as $\mathbb{N}$-indexed unions of sets in $\bigcup_{\tau<\eta} \mathbf{\Pi}^0_\tau(X)$.

Another way of putting this definition is the following: $\mathbf{\Sigma}^0_1(X)$ are open sets; $\mathbf{\Pi}^0_\eta(X)$ are closed sets; and for any $\eta > 1$ we have:

$$\mathbf{\Sigma}^0_\eta(X) = \left\{ \bigcup_{n\in\mathbb{N}} A_n \mid \forall n \in \mathbb{N}.\ A_n \in \bigcup_{\tau<\eta} \mathbf{\Pi}^0_\tau(X) \right\}$$

$$\mathbf{\Pi}^0_\eta(X) = \left\{ \bigcap_{n\in\mathbb{N}} A_n \mid \forall n \in \mathbb{N}.\ A_n \in \bigcup_{\tau<\eta} \mathbf{\Sigma}^0_\tau(X) \right\}$$

Example 2.1. *The class of G_δ sets in X coincides with $\mathbf{\Pi}^0_2(X)$. A countable union of G_δ sets is always a member of $\mathbf{\Sigma}^0_3(X)$. If $A_n \in \mathbf{\Sigma}^0_{2n+7}(X)$ for $n = 0, 1, \ldots$ then $\bigcap_{n\in\mathbb{N}} A_n \in \mathbf{\Pi}^0_\omega(X)$.*

Now we put $\mathbf{\Delta}^0_\eta(X) = \mathbf{\Sigma}^0_\eta(X) \cap \mathbf{\Pi}^0_\eta(X)$, for instance $\mathbf{\Delta}^0_1(X)$ is the family of clopens in X.

Exercise 4. *Prove that $\mathbf{\Sigma}^0_\eta(X) \subseteq \mathbf{\Pi}^0_{\eta+1}(X)$ and symmetrically $\mathbf{\Pi}^0_\eta(X) \subseteq \mathbf{\Sigma}^0_{\eta+1}(X)$.*

The inclusions between the consecutive classes are depicted as follows:

$$\begin{array}{ccccccccc}
 & & \mathbf{\Sigma}^0_1 & & \mathbf{\Sigma}^0_2 & & \mathbf{\Sigma}^0_3 & & \\
 & / & & \backslash \quad / & & \backslash \quad / & & & \\
\mathbf{\Delta}^0_1 & & & \mathbf{\Delta}^0_2 & & \mathbf{\Delta}^0_3 & & & \cdots \\
 & \backslash & & / \quad \backslash & & / \quad \backslash & & & \\
 & & \mathbf{\Pi}^0_1 & & \mathbf{\Pi}^0_2 & & \mathbf{\Pi}^0_3 & &
\end{array}$$

Exercise 5. *Prove that each class $\mathbf{\Sigma}^0_\eta(X)$ is closed under countable unions. Analogously, show that each class $\mathbf{\Pi}^0_\eta(X)$ is closed under countable intersections.*

Recall that ω_1 is the minimal ordinal number that is not countable (as the set of all smaller ordinals).

Exercise 6. *Prove that if $0 \leqslant \eta_0 \leqslant \eta_1 \leqslant \ldots < \omega_1$ then there exists a countable ordinal $\eta < \omega_1$ such that $\forall n \in \mathbb{N}.\ \eta_n < \eta$.*

Define

$$
\begin{aligned}
B_\Delta(X) &= \bigcup_{\eta<\omega_1} \boldsymbol{\Delta}^0_\eta(X)\\
B_\Sigma(X) &= \bigcup_{\eta<\omega_1} \boldsymbol{\Sigma}^0_\eta(X)\\
B_\Pi(X) &= \bigcup_{\eta<\omega_1} \boldsymbol{\Pi}^0_\eta(X)
\end{aligned}
$$

Exercise 7. *Prove that* $B_\Delta(X) = B_\Sigma(X) = B_\Pi(X) = \mathcal{B}(X)$.

Finally, we use the notions $\boldsymbol{\Sigma}^0_\eta$, $\boldsymbol{\Pi}^0_\eta$, etc without the parameter X as the classes of sets from all the possible Polish topological spaces. More formally, we assume that $A \in \boldsymbol{\Sigma}^0_7$ implicitly means that a Polish topological space X is known from the context, $A \subseteq X$, and $A \in \boldsymbol{\Sigma}^0_7(X)$.

2.2 Boldface pointclasses

For the sake of these exercises, we will say that Γ (or $\Gamma(X)$ for all Polish spaces X, see the discussion above) is a *boldface pointclass*[3] if whenever $f\colon X \to Y$ is a continuous function between two Polish spaces and $A \in \Gamma(Y)$ then $f^{-1}(A) \in \Gamma(X)$.

Exercise 8. *Prove that all the classes* $\boldsymbol{\Sigma}^0_\eta$, $\boldsymbol{\Pi}^0_\eta$, $\boldsymbol{\Delta}^0_\eta$ *are boldface pointclasses.*

Definition 2.2. *By* $\boldsymbol{\Pi}^1_1$ *we denote the class of complements of analytic (i.e.* $\boldsymbol{\Sigma}^1_1$*) sets. We call these sets coanalytic.*

Exercise 9. *Prove that additionally the classes of analytic sets* $\boldsymbol{\Sigma}^1_1$ *and co-analytic sets* $\boldsymbol{\Pi}^1_1$ *are boldface pointclass.*

By Γ^{c} we denote the pointclass of complements of sets in Γ.

By the definition of the Borel hierarchy, for every countable ordinal $\eta \geqslant 1$ we have $\left(\boldsymbol{\Sigma}^0_\eta\right)^{\mathrm{c}} = \boldsymbol{\Pi}^0_\eta$ and $\left(\boldsymbol{\Pi}^0_\eta\right)^{\mathrm{c}} = \boldsymbol{\Sigma}^0_\eta$.

Exercise 10. *Prove that if* Γ *is a boldface pointclass then* Γ^{c} *is also a boldface pointclass.*

[3] *Pointclass* means that it is a collection of sets of points, where each point is a member of a Polish topological space (known from the context).

2.3 Universal sets

Let Γ be a boldface pointclass. We say that a set $U \subseteq Y \times X$ is *universal for* Γ if $U \in \Gamma(Y \times X)$ and

$$\Gamma(X) = \{U_y \mid y \in Y\},$$

where the sections U_y are defined as $\{x \in X \mid (y, x) \in U\}$.

Remark 2.3. *A universal program is a computer program $U(p, x)$ with two parameters:*

- *p is a code of another computer program $P(x)$*
- *x is an argument (can be a natural number)*

such that the execution of $U(p, x)$ with arguments p, x returns as a result a value y if and only if the execution of the program $P(x)$ with the argument x returns y.

In other words, $U(p, .)$ behaves exactly like the program P whose code is p.

Exercise 11. *Take any at most countable set Z. Show that the following set is universal for $\mathbf{\Pi}^0_1$:*

$$U_F = \left\{(T, \alpha) \in \mathrm{Tr}_Z \times Z^\omega \mid \forall n \in \mathbb{N}.\ \alpha{\upharpoonright}_n \in T\right\}.$$

Exercise 12. *Let $U \subseteq Y \times X$ be a universal set for a boldface pointclass Γ. Construct a universal set for Γ^{c}.*

Exercise 13. *Prove that if Γ is a boldface pointclass that has a universal set*[4] *$U \subseteq X \times X$ then $\Gamma \neq \Gamma^{\mathrm{c}}$.*

Remark 2.4. *One may see it as a Descriptive-Set-Theoretic variant of undecidability of the halting problem (see Remark 2.3). Btw. possibly (?) the DST result was proven first.*

Exercise 14. *Given a universal set $U \subseteq \mathbb{N}^{\mathbb{N}} \times X$ for $\mathbf{\Sigma}^0_\eta$ construct a universal set $U' \subseteq \mathbb{N}^{\mathbb{N}} \times X$ for $\mathbf{\Pi}^0_{\eta+1}$.*

Exercise 15. *Construct a universal set $U \subseteq \mathbb{N}^{\mathbb{N}} \times X$ for $\mathbf{\Sigma}^0_\omega$, assuming that for every $n \in \mathbb{N}$ there exists a universal set $U_n \subseteq \mathbb{N}^{\mathbb{N}} \times X$ for $\mathbf{\Pi}^0_n$.*

[4] For technical reasons we assume that $X = Y$ here.

The above construction works in the same way for other limit ordinal numbers. Therefore, we obtain the following theorem.

Theorem 2.5. *For every at most countable set Z and every countable ordinal $\eta \geqslant 1$ there exist sets $U_\Sigma \subseteq \mathbb{N}^\mathbb{N} \times Z^\omega$ and $U_\Pi \subseteq \mathbb{N}^\mathbb{N} \times Z^\omega$ that are universal for $\mathbf{\Sigma}^0_\eta$ and $\mathbf{\Pi}^0_\eta$ respectively.*

Exercise 16. *Borel hierarchy is strict, in the sense that for every countable ordinal $\eta \geqslant 1$ we have $\mathbf{\Sigma}^0_\eta \subsetneq \mathbf{\Pi}^0_{\eta+1}$ and $\mathbf{\Pi}^0_\eta \subsetneq \mathbf{\Sigma}^0_{\eta+1}$.*

Exercise 17. *Is there any universal set $U \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ for $\mathbf{\Delta}^0_3$?*

3 New homework

Exercise 18. *Solve Exercise 13 in a written form.*
You can use the Cantor's diagonal argument from page 13 of the lecture notes.

Exercise 19 ($\star$)**.** *Prove that the following set is universal for $\mathbf{\Sigma}^1_1$:*

$$\big\{(T,\alpha) \in \mathrm{Tr}_{\mathbb{N}\times\mathbb{N}} \times \mathbb{N}^\mathbb{N} \mid \exists \beta \in \mathbb{N}^\mathbb{N}.\ (\alpha,\beta) \in [T]\big\}.$$

You may follow the idea (left as an exercise) from pages 11–12 of the lecture notes.

[[in the above formulation we silently use the homeomorphism between $\mathbb{N}^\omega \times \mathbb{N}^\omega$ and $(\mathbb{N}\times\mathbb{N})^\omega$]]

4 Hints

Hint to Exercise 3 Take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \pi_X(F)$ that is convergent to $x \in X$. We need to show that $x \in \pi_X(F)$. Each of the elements x_n comes with a witness y_n such that $(x_n, y_n) \in F$. Consider a converging subsequence of y_n (w.l.o.g. we can assume that this is the whole sequence) whose limit is y. Then, $(x, y) \in F$ and therefore $x \in \pi_X(F)$.

Hint to Exercise 4 One can take a countable union or intersection of infinitely many copies of the same set...

Hint to Exercise 5 A countable union of countable unions is a countable union.

Hint to Exercise 6 Each ordinal number is the set of all smaller ordinals. Therefore, it makes sense to consider the union $x = \bigcup_{n\in\mathbb{N}} \eta_n$. Clearly x is at most countable as a set. Moreover, x is an ordinal number, because it contains only ordinal numbers and if $\tau \in x$ and $\tau' < \tau$ then also $\tau' \in x$. Thus, $x < \omega_1$ and taking $\eta = x + 1$ (for safety) we have the thesis.

Hint to Exercise 7 The equalities $B_\Delta(X) = B_\Sigma(X) = B_\Pi(X)$ follow from the inclusions between the classes. Therefore, it is enough to show that, say $B_\Sigma(X) = \mathcal{B}(X)$. The inclusion $B_\Sigma(X) \subseteq \mathcal{B}(X)$ is easily proven inductively. For the opposite inclusion it is enough to show that $B_\Sigma(X)$ is a σ-algebra. Thus, one can show that $B_\Sigma(X)$ is closed under countable unions and $B_\Pi(X)$ is closed under countable intersections. Take any countable family of sets $(A_n)_{n\in\mathbb{N}} \subseteq B_\Sigma(X)$. Observe, that for each n there exists $\eta_n < \omega_1$ such that $A_n \in \mathbf{\Sigma}^0_{\eta_n}(X)$. Apply Exercise 6 to observe that there exists $\eta < \omega_1$ such that $\forall n \in \mathbb{N}.\ A_n \in \mathbf{\Sigma}^0_\eta$. Then $\bigcup_{n\in\mathbb{N}} A_n \in \mathbf{\Sigma}^0_\eta(X)$ as well.

Hint to Exercise 8 One can prove it by induction: it holds for open and closed sets. Moreover, preimages go well with countable Boolean operations.

Hint to Exercise 9 Take $F \subseteq Y \times Z$ that is closed and $A = \pi_Y(F)$ is an analytic set in Y. Let $f\colon X \to Y$ be continuous. Let

$$F' = \{(x, z) \in X \times Z \mid (f(x), z) \in F\}.$$

Check that F' is closed and $\pi_X(F') = f^{-1}(A)$.

Hint to Exercise 10 Note that $f^{-1}(A^{\mathrm{c}}) = \big(f^{-1}(A)\big)^{\mathrm{c}}$, where the complements $(.)^{\mathrm{c}}$ are taken in the respective spaces.

Hint to Exercise 11 Recall that a set $A \subseteq Z^\omega$ is closed if and only if $A = [T]$ for some tree $T \in \mathrm{T}_Z$.

Hint to Exercise 12 Take $U' = U^{\mathrm{c}}$.

Hint to Exercise 14 Notice that $\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}$ is homeomorphic with $\mathbb{N}^{\mathbb{N}}$. Thus, one can consider

$$U' = \left\{\left((\alpha_0, \alpha_1, \ldots), x\right) \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times X \mid \forall n \in \mathbb{N}.\ (\alpha_n, x) \in U\right\}.$$

Hint to Exercise 15 Do essentially the same as above.

Hint to Exercise 16 If $\boldsymbol{\Sigma}^0_\eta = \boldsymbol{\Pi}^0_{\eta+1}$ then $\boldsymbol{\Sigma}^0_{\eta+1} = \boldsymbol{\Pi}^0_{\eta+1}$, which is in contradiction with Exercise 13 and Theorem 2.5.

Hint to Exercise 17 Directly from the definition we see that $\left(\boldsymbol{\Delta}^0_\eta\right)^{\mathrm{c}} = \boldsymbol{\Delta}^0_\eta$. Therefore, Exercise 13 implies that there is no such universal set.