This file provides materials of the tutorial that was expected on 12th March. Please study this file carefully before the next lecture (19th March). In case of any questions, please contact me by e-mail. I'm willing to explain as much as needed either by mail or some more online tool (shared whiteboard etc).

The material is divided into four parts: first, solutions of the previous homework problems are given. Then the core flow of the tutorial comes, with all the problems posed and discussed. Then, the new homework problems are given. The final part contains hints to some less clear exercises. The aim of that division is to motivate you to work autonomously and not just read the solutions straight ahead.

The deadline for the new homework is again this Friday 24:00.

1 Solutions of the homework problems

Definition 1.1. *Take a polish topological space X. Let $K(X)$ be the space of all compact subsets of X, equipped with the* Vietoris topology *generated by*

$$\{K \in K(X) \mid K \subseteq U\}$$
$$\{K \in K(X) \mid K \cap U \neq \varnothing\},$$

for U open in X.

Definition 1.2. *Define the* Hausdorff metric *on $K(X)$ by*

$$\begin{aligned} d_H(K,L) &= 0 && \text{if } K = L = \varnothing \\ &= 1 && \text{if exactly one of } K,\ L \text{ is } \varnothing \\ &= \max\{\delta(K,L), \delta(L,K)\} && \text{if } K, L \neq \varnothing \end{aligned}$$

where

$$\delta(K,L) = \max_{x \in K} d(x,L),$$

and

$$d(x,L) = \inf_{y \in L} d(x,y).$$

Exercise 1. *Show that Hausdorff metric is a metric and that it is compatible with Vietoris topology (i.e. generates the same topology).*
Moreover, show that if X is separable, so is $K(X)$.
Similarly, show that if X is compact then so is $K(X)$.

The only problematic part was the last one: if X is compact then $K(X)$ is also compact. For that, it is enough to show the following two claims. These proofs are based on a solution of the homework by Tomasz Jabłczyński.

Claim 1.3. *If X is complete then $K(X)$ is also complete.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $K(X)$. Let

$$L = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}\Big(\bigcup_{k>n} K_k\Big).$$

Clearly L is closed as an intersection of closed sets. Since X is compact, $L \in K(X)$. We claim that L is the limit of the sequence $(K_n)_{n\in\mathbb{N}}$. Take $\varepsilon > 0$ and let N be such that for $n, m \geqslant N$ we have $d_H(K_n, K_m) < \varepsilon/2$. We will show that $d_H(L, K_N) < \varepsilon$.

Take $x \in L$. It means that $x \in \mathrm{Cl}(\bigcup_{k>N} K_k)$ and therefore there exists $k > N$ such that $B(x, \varepsilon/2) \cap K_k \neq \varnothing$. Thus, $d(x, K_k) < \varepsilon/2$. Since also $d_H(K_k, K_N) < \varepsilon/2$, we obtain that $d(x, K)N) < \varepsilon$. Since x was chosen arbitrarily in L, we know that $\delta(L, K_N) < \varepsilon$.

Now take $y \in K_N$ and let $x_n \in B(y, \varepsilon/2) \cap K_n$ for $n \geqslant N$ — these points exists, because of the assumption that $d_H(K_n, K_m) < \varepsilon/2$ for $n, m \geqslant N$. As X is compact, the sequence x_n has a subsequence that converges to a point x. It is easy to see that $x \in L$ and $d(y, x) \leqslant \varepsilon/2 < \varepsilon$. By the arbitrary choice of y we know that $\delta(K_N, L) < \varepsilon$. Notice that we additionally show here that if infinitely many of $(K_n)_{n\in\mathbb{N}}$ are non-empty then also L is non-empty.

Summing up, $d_H(L, K_N) < \varepsilon$ and therefore L is a limit of a subsequence of the sequence $(K_n)_{n\in\mathbb{N}}$. But as this is a Cauchy sequence, $\lim_{n\to\infty} K_n = L$. ■

Claim 1.4. *If X is compact then $K(X)$ is totally bounded.*

Proof. Take $\varepsilon > 0$ and let $A \subseteq X$ be a finite set such that $\bigcup_{x\in A} B(x, \varepsilon) = X$. Take $K \in K(X)$. Let $A_K = \{x \in A \mid K \cap B(x, \varepsilon) \neq \varnothing\}$. Clearly A_K is compact as a finite set. Moreover, $\delta(K, A_K) < \varepsilon$ and $\delta(A_K, K) < \varepsilon$. Thus, $K \in B(A_K, \varepsilon)$. ■

Exercise 2 ($\star$)**.** *Consider X at most countable, with discrete topology.*

(a) *Show that the following conditions are equivalent for a set $A \subseteq X^\omega$*

1. *A is G_δ;*

2. there exists a subset $R \subseteq X^{<\omega}$ such that

$$A = \left\{\alpha \in X^\omega \mid \alpha{\restriction}_n \in R \text{ for infinitely many } n\right\};$$

3. there exists a continuous function $f\colon X^\omega \to 2^\omega$ such that $f^{-1}(G) = A$, for $G = \{\alpha \in 2^\omega \mid \alpha$ has infinitely many $1s\}$.

(b) *Take a set A that is $\mathbf{\Delta}^0_2$ in X^ω. Show that for every $s \in X^{<\omega}$ there exists $r \in X^{<\omega}$ such that $s \preceq r$ and N_r is either contained or disjoint from A.*

The following argument is based on a solution by Damian Gładkowski.

Implication (1) $\Rightarrow$ (2) Consider $A = \bigcap_{n\in\mathbb{N}} U_n$ with a sequence of open sets U_n. Without loss of generality we can assume that the sequence $(U_n)_{n\in\mathbb{N}}$ is descending (one can take $U'_n = U_0 \cap \ldots \cap U_n$). We can represent each U_n as a disjoint union of basic open sets $U_n = \bigcup_{s\in R_n} N_s$ for R_n defined by

$$R_n = \left\{s \in X^{<\omega} \mid N_s \subseteq U_n \wedge \left(s = \epsilon \vee N_{s\restriction_{|s|-1}} \nsubseteq U_n\right)\right\}.$$

Define $T = \{s \in X^{<\omega} \mid s$ belongs to infinitely many sets $R_n\}$. We claim that

$$R = \bigcup_{n\in\mathbb{N}} R_n \cup \{r \in X^{<\omega} \mid \exists s \in T.\ s \preceq r\}$$

is the desired set from the statement.

Notice that since the family U_n is descending, $\alpha \in A = \bigcap_{n\in\mathbb{N}} U_n$ if and only if α belongs to infinitely many U_n.

First take $\alpha \in X^\omega$ such that for infinitely many n we have $\alpha{\restriction}_n \in R$. The first case is that some prefix $s \preceq \alpha$ belongs to T. In that case α belongs to infinitely many U_n and therefore $\alpha \in A$. Assume contrarily that no such prefix exists. In that case infinitely many prefixes of α belong to the union $\bigcup_{n\in\mathbb{N}} R_n$. However, members of each set R_n separately are $\preceq$-incomparable. Therefore, the prefixes of α belong to infinitely many sets R_n. Thus, α belongs to infinitely many of the sets U_n and $\alpha \in A$.

Now assume that $\alpha \in A$. It means that for each n there exists a prefix $s_n \preceq \alpha$ such that $s_n \in R_n$. (its length may not equal n). If the set $\{s_n \mid n \in \mathbb{N}\}$ is finite then it intersects T and therefore almost all prefixes of α belong to R. Otherwise, $\{s_n \mid n \in \mathbb{N}\}$ is infinite. But clearly $\{s_n \mid n \in \mathbb{N}\} \subseteq R$.

Implication (2) ⇒ (3) Take R from the previous exercise and define $f\colon X^\omega \to 2^\omega$ by $f(\alpha)(n) = \mathbf{1}_R(\alpha\restriction_n)$, where $\mathbf{1}_R$ is the indicator of the set R. The function f is continuous coordinate-wise so it is continuous. Clearly, $\alpha \in A$ if and only if $f(\alpha) \in G$.

Implication (3) ⇒ (1) G_δ sets are closed under continuous preimages (because open sets are and countable Boolean operations are). Therefore, it is enough to see that G is G_δ itself. However, the complement of G is countable, because it is the set of sequences of the form $s \cdot 0^\omega$ for $s \in 2^{<\omega}$ and 0^ω the sequence of only-zeros. Therefore, the complement of G is a countable union of closed sets and therefore G is a countable intersection of open sets.

Argument for (b) [it can also be proved directly by Baire Category Method]

Take a set A that is $\mathbf{\Delta}^0_2$ and let R and R' be sets representing A and $X^\omega - A$ respectively as in (1).

Assume contrarily and take $r_0 \in X^{<\omega}$ such that no $r \succeq r_0$ satisfies the given property. It means that for every $s \in X^{<\omega}$ satisfying $r_0 \preceq s$ there exist $r, r' \succeq s$ such that $r \in R$ and $r'\ inR$. This allows us to inductively define $r_0 \prec s_0 \prec s_1 \prec s_2 \prec \ldots$ such that $s_{2n} \in R$ and $s_{2n+1} \in R'$. Let $\alpha \in X^\omega$ be the unique point such that $s_n \prec \alpha$ for all n. Then α has infinitely many prefixes both in R and R' so $\alpha \in A \cap (X - A)$. A contradiction.

2 New material

2.1 Borel sets and isomorphisms

During the lecture the notion of *Borel sets* (the smallest σ-algebra containing all open sets) was defined.

Exercise 3. *Let X be a Polish space and $F \subseteq X$ closed. Prove that F is Borel.*

A function $f\colon X \to Y$ is Borel if the preimages $f^{-1}(U) \subseteq X$ of open sets $U \subseteq Y$ are all Borel in X.

Exercise 4. *Show that a function $f\colon X \to Y$ is Borel if and only if the preimages of all Borel sets in Y are Borel in X.*

A *Borel isomorphism* is a bijection that is both Borel and its inverse is also Borel.

Exercise 5. *Take two functions $f\colon X \to Y$ and $g\colon Y \to X$. Assume that the ranges $f(X)$ and $g(Y)$ are Borel. Assume moreover that both functions are Borel isomorphisms between their domains and ranges. Under these assumptions show that there exists a Borel isomorphism between X and Y.*

The above fact was used to show that every two uncountable Polish spaces are Borel isomorphic.

Exercise 6. *Fix a sequence of Polish topologies $(\tau_n)_{n\in\mathbb{N}}$ on a Polish topological space (X,τ). Assume that $\tau \subseteq \tau_n \subseteq \mathcal{B}(X,\tau)$, where the inclusion is understood in terms of families of open sets and $\mathcal{B}(X,\tau)$ is the family of Borel subsets of X w.r.t. τ.*

Let τ_∞ be the topology on X generated by $\bigcup_{n\in\mathbb{N}} \tau_n$. Show that this topology is Polish and $\mathcal{B}(X,\tau_\infty) = \mathcal{B}(X,\tau)$.

The idea given in the lecture was to consider spaces $X_n = (X,\tau_n)$ for $n \in \mathbb{N}$ and the diagonal map $\phi\colon X \to \prod_{n\in\mathbb{N}} X_n$ given by

$$\phi(x) = (x, x, \ldots).$$

The claim of the theorem follows from the observation that ϕ is a homeomorphism between X and the diagonal

$$\phi(X) = \big\{(x, x, \ldots) \mid x \in X\big\} \subseteq \prod_{n\in\mathbb{N}} X_n.$$

Based on this fact, the following important result was obtained:

Theorem 2.1. *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of Borel subsets of a Polish topological space (X,τ) then there exists a Polish topology $\tau' \supseteq \tau$ on X such that all the sets A_n are clopen in τ'.*

2.2 Baire Category Method

Recall the following statement of Baire theorem.

Theorem 2.2. *Let X be a (non-empty) Polish topological space and sets $(U_n)_{n\in\mathbb{N}}$ be open and dense in X. Then the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is also dense in X (in particular non-empty).*

Exercise 7. *Prove the above theorem.*

Exercise 8. *Show that if $(G_n)_{n\in\mathbb{N}}$ is a sequence of dense G_δ sets (in a Polish topological space X) then the intersection $\bigcap_{n\in\mathbb{N}} G_n$ is a dense G_δ set.*

Exercise 9. *Prove that the set of rational numbers $\mathbb{Q}$ with the topology inherited from $\mathbb{R}$ is not a Polish topological space.*

Exercise from the previous lecture:

Let X be an uncountable perfect Polish topological space. Let $(B_n)_{n\in\mathbb{N}}$ be a fixed countable basis of X. For $n \in \mathbb{N}$ define F_n as $\mathrm{Cl}(B_n) - B_n$, where $\mathrm{Cl}(A)$ is the intersection of all closed sets containing A and $\mathrm{Int}(A)$ is the union of all open sets contained in A.

Exercise 10. *Prove that $X - \bigcup_{n\in\mathbb{N}} F_n$ is a dense G_δ subset of X. Moreover, show that this space (with the topology inherited from X) is zero-dimensional.*

Based on that, it was shown that every uncountable Polish topological space contains a dense G_δ that is isomorphic with $\mathbb{N}^\omega$.

Another important thing that you should remember is:

Theorem 2.3. *If X is a Polish topological space and $A \subseteq X$ then A (with the induced topology) is a Polish space if and only if A is a G_δ subset of X.*

2.3 Analytic sets

A set $A \subseteq X$ of a Polish space X is *analytic* if it is a continuous image of a Polish topological space. The family of all analytic subsets of X is denoted $\mathbf{\Sigma}^1_1(X)$.

Exercise 11. *Show that if X, Y are Polish and $A \subseteq X \times Y$ is Borel then the projection $\pi_X(A)$ is analytic in X.*

Exercise 12. *Show the reverse, that if $A \subseteq Y$ is analytic (with $f\colon X \to Y$ witnessing that by $f(X) = A$) then A is a projection of a* **closed** *subset of some product space $Y \times Z$ with Z Polish.*

Exercise 13. *Take X at most countable. Consider the space of trees over X trees Tr_X as a subset of $2^{X^{<\omega}}$. Clearly $2^{X^{<\omega}}$ with the product topology is a Polish space. Show that Tr_X is a closed subset of this space. Therefore, it's also a Polish space.*

Prove that the set of ill-founded *trees over* ω

$$\mathrm{IF} = \{T \in \mathrm{Tr}_\omega \mid [T] \neq \varnothing\}$$

i.e. trees with at least one infinite branch is an analytic subset of Tr*.*

3 New homework

Exercise 14. *Let* Ord *be the subset of* $2^{\mathbb{N}\times\mathbb{N}}$ *that contains* $o \in 2^{\mathbb{N}\times\mathbb{N}}$ *(treated as a subset of* $\mathbb{N}\times\mathbb{N}$*) if and only if* o *is a linear order. Show that* Ord *is a closed subset of* $2^{\mathbb{N}\times\mathbb{N}}$*.*

Consider $\mathrm{IO} \subseteq \mathrm{Ord}$ *that contains* o *if and only if* o *is not well-founded (it contains a subset that has no minimal element). Show that* IO *is an analytic set* $\mathbf{\Sigma}^1_1$*.*

Exercise 15 ($\star$)**.** *Show that the set of trees over* 2 *with an infinite branch (i.e. ill-founded binary trees in* Tr_2*) is a Borel subset of* Tr_2*.*

4 Hints

Hint to Exercise 3 Try to represent F as a countable intersection of open sets.

Hint to Exercise 4 One implication is obvious. For the other, take $f\colon X \to Y$ such that preimages of open sets are Borel. Consider the family $\mathcal{C}$ of subsets of Y defined by: $C \in \mathcal{C}$ if $f^{-1}(C)$ is Borel in X. Clearly the family $\mathcal{C}$ contains all the open sets of Y. Check that is is closed under countable Boolean operations. Thus, by the definition $\mathcal{B}(Y)$ is contained in $\mathcal{C}$.

Hint to Exercise 5 As suggested in the lecture, retry the standard (constructive) proof of Cantor–Bernstein theorem, see e.g. proof 1 in `https://pl.wikipedia.org/wiki/Twierdzenie_Cantora-Bernsteina-Schr%C3%B6dera`

Notice that all the involved sets are Borel and the respective functions on them are also Borel. Argue that you can glue a countable family of Borel function into one Borel function.

Hint to Exercise 6 Seems to be just a simple check.

Hint to Exercise 7 Take a ball B in X. We need to show that the intersection $\bigcap_{n\in\mathbb{N}} U_n$ intersects that ball. We will define a descending sequence of balls B_n such that $\mathrm{Cl}(B_{n+1}) \subseteq B_n \subseteq B$. We begin with a ball B_0 of radius half of B. Now for $n = 0, \ldots$ the set U_n is dense and open, so it intersects B_n, so there exists a ball B_{n+1} that (with its closure) is contained in $U_n \cap B_n$.

This way (by using completeness of the space) we get a point $x \in \bigcap_{n\in\mathbb{N}} B_n$ that belongs to both $\bigcap_{n\in\mathbb{N}} U_n$ and B.

Hint to Exercise 8 Write each of G_n as an intersection of $U_{n,k}$ for $k \in \mathbb{N}$ and consider the doubly-indexed family $(U_{n,k})_{n,k\in\mathbb{N}}$.

Hint to Exercise 9 Assume contrarily and observe that for each rational number q the set $U_q = \mathbb{Q} - \{q\}$ is open and dense in $\mathbb{Q}$. Their intersection is however empty. . . :(

Hint to Exercise 10 For the first claim it is enough to observe that each F_n is closed and its complement is dense (easy to check). Therefore, the complements U_n of F_n are open and dense and therefore their intersection is a dense G_δ.

For zero-dimensionality, one observes that $B_n - F_n$ is a basis of the new space that consists of clopens only.

Hint to Exercise 11 We already know that every Borel set is a continuous image of a Polish space (is proved in the last lecture, page 9 item (iii)). Projections are always continuous and a composition of continuous functions is continuous.

Hint to Exercise 12 Take a continuous function $f\colon X \to Y$ with $f(X) = A$. Let $Z = X$ and let $F \subseteq Y \times X$ be the set of pairs $\{(f(x), x) \mid x \in X\}$. As f is continuous, F is closed.

Hint to Exercise 13 Consider the space $\mathrm{Tr}_\omega \times \mathbb{N}^\omega$ and the set $\{(T, \alpha) \mid \alpha \in [T]\}$.