Additional Homework

This homework is designed in such a way to recall certain notions that were previously discussed. Therefore, please solve the exercises in a standalone way: do not invoke facts known from the lecture or exercises but provide direct arguments based purely on the respective definitions. If you hesitate if something can be invoked and used as a black-box, please ask.

Recall that Tr_ω is a subset of the Polish space $2^{\omega^{<\omega}}$ containing those subsets of $\omega^{<\omega}$ that are prefix-closed. Recall the notion of $[t]$ for a tree t. Given $\alpha \in X^\omega$ and $\beta \in Y^\omega$ by $\langle \alpha, \beta \rangle \in (X \times Y)^\omega$ we denote the natural zip of the two sequences.

Recall the notion of Γ-universal sets. Recall and use certain Tr_X-universal set for $\mathbf{\Pi}^0_1(X^\omega)$ — you don't need to prove that it is universal if it was already done.

Exercise 1 ($\geqslant 0.5$ points)**.** *Check that the following set is* $\mathrm{Tr}_{\omega\times\omega}$*-universal for analytic sets:*

$$U \stackrel{def}{=} \big\{(t, \alpha) \in \mathrm{Tr}_{\omega\times\omega} \times \mathbb{N}^\omega \mid \exists \beta \in \mathbb{N}^\omega .\ \langle \alpha, \beta \rangle \in [t]\big\}.$$

Exercise 2 ($\geqslant 0.5$ points)**.** *Take two Polish spaces X and Y and a boldface pointclass Γ. Assume that $f\colon Y \to X$ is continuous sujection of Y onto X. Show that if $U \subseteq X \times Y$ is an X-universal set for Γ, i.e. $\Gamma(Y) = \{U_x \mid x \in X\}$ then $\Gamma \neq \Gamma^{\mathrm{c}}$.*

Entail that the set U above is not coanalytic (in particular not Borel).

Exercise 3 ($\geqslant 0.5$ points)**.** *Provide a complete stand-alone argument that if f is continuous and $B \subseteq X$ is coanalytic then $f^{-1}(B)$ is also coanalytic.*

Recall that $\mathrm{IF} \subseteq \mathrm{Tr}_\omega$ is the set of trees that have some infinite branch. We know that $\mathrm{IF} \in \mathbf{\Sigma}^1_1$.

Exercise 4. *Show that* IF *is not coanalytic, because there exists a continuous function f such that $f^{-1}(\mathrm{IF}) = U$ and by the above exercise U is not coanalytic.*

Exercise 5 ($\star$)**.** *Give an example of a coanalytic relation $B \subseteq \mathrm{Tr}_\omega \times Y$ for some Polish space Y such that $\pi_{\mathrm{Tr}_\omega}(B) = \mathrm{IF}$ and the sections of that relation are at most singletons: for every $t \in \mathrm{Tr}_\omega$ we have $|B_x| \leqslant 1$.*

The above exercise shows that the first exercise from 7th May does not extend to coanalytic sets: they may have small sections and provide non-coanalytic projections. In fact every coanalytic relation R admits a coanalytic subset $F \subseteq R$ that has singleton sections and the same projection as R...