Definability of choice over scattered trees in MSO

Michał Skrzypczak

Highlights 2015
Prague

Logic:

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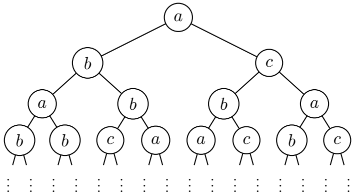

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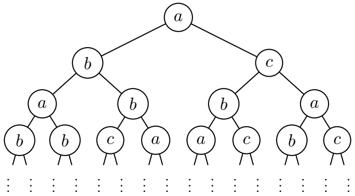

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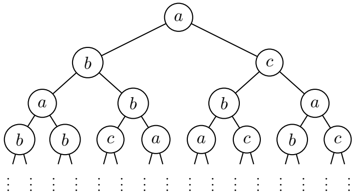

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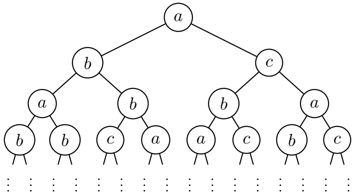

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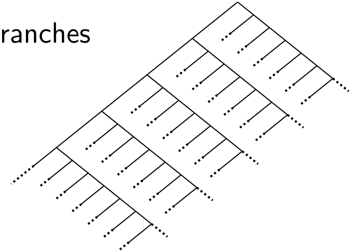

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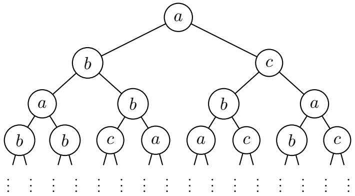

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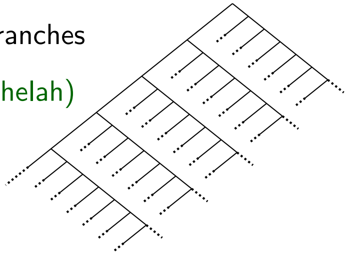

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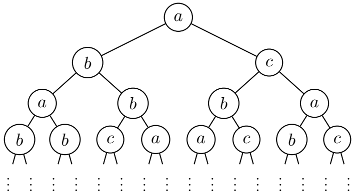

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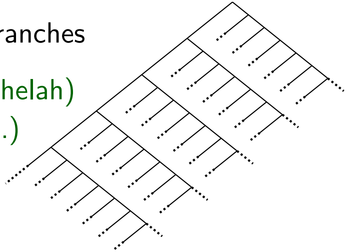

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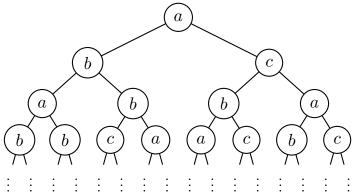

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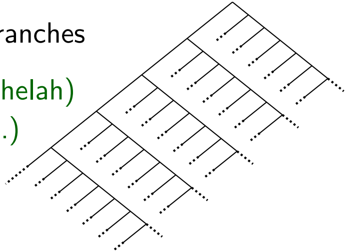

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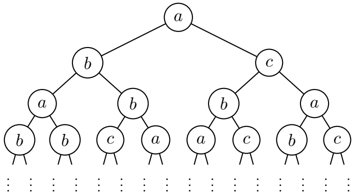

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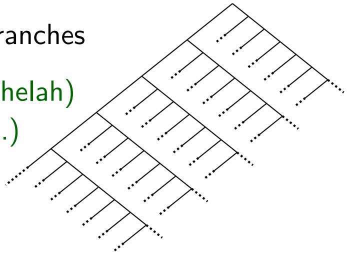

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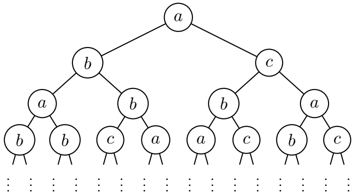

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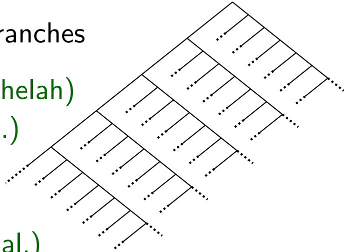

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Theorem (Bilkowski, S. [2013])
The above conjecture implies an effective characterisation of bi-unambiguous languages of complete trees.

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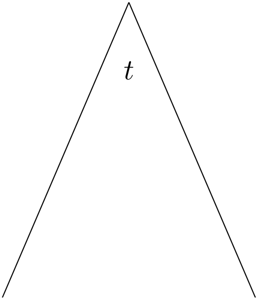

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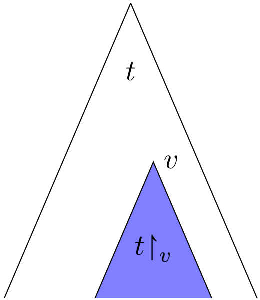

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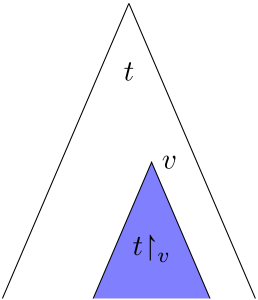

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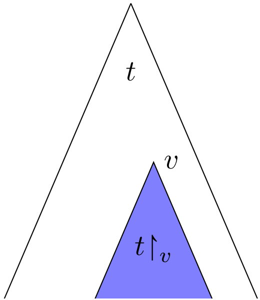
[if there exists $\alpha\colon \mathrm{Trees} \to H\ldots$]



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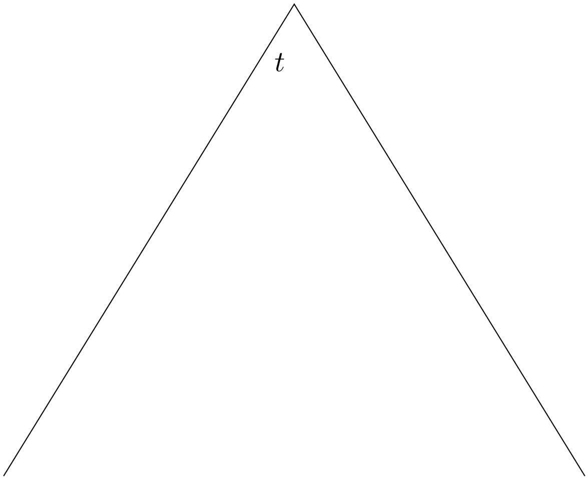

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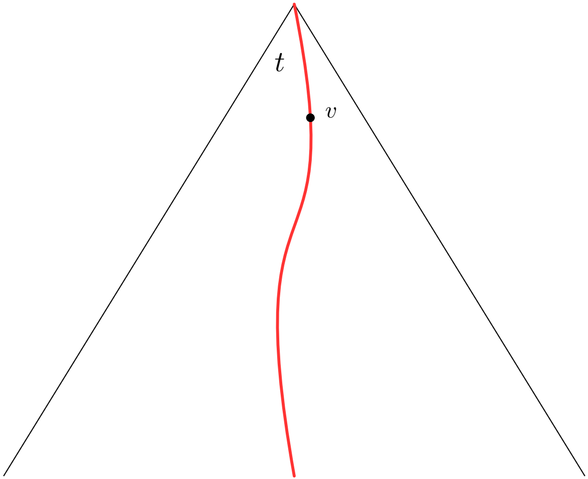

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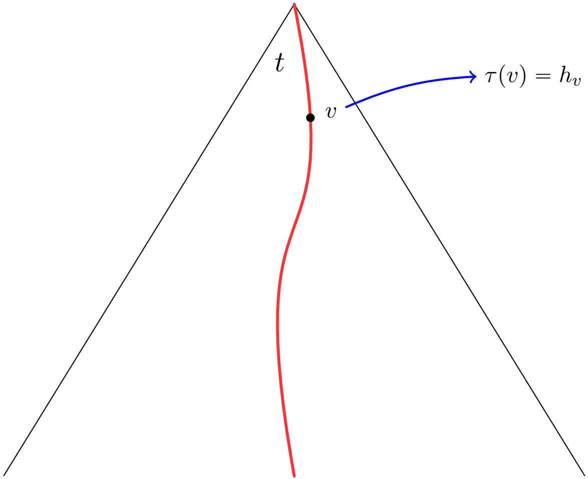

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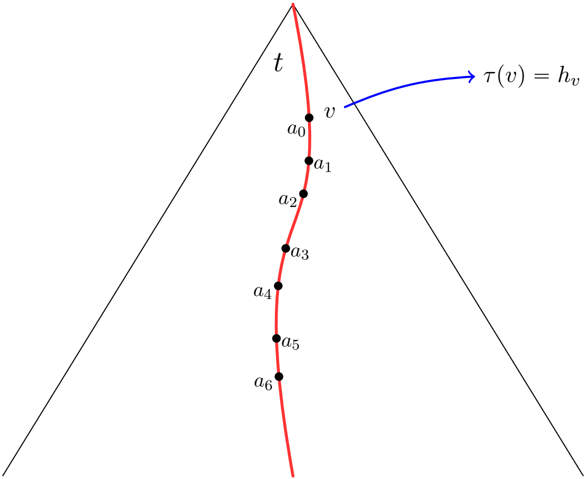

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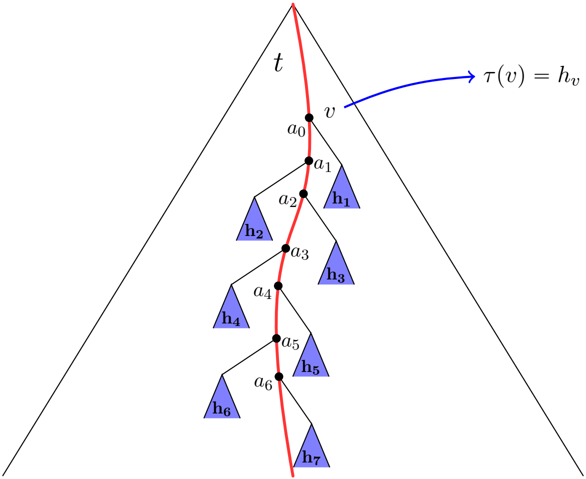

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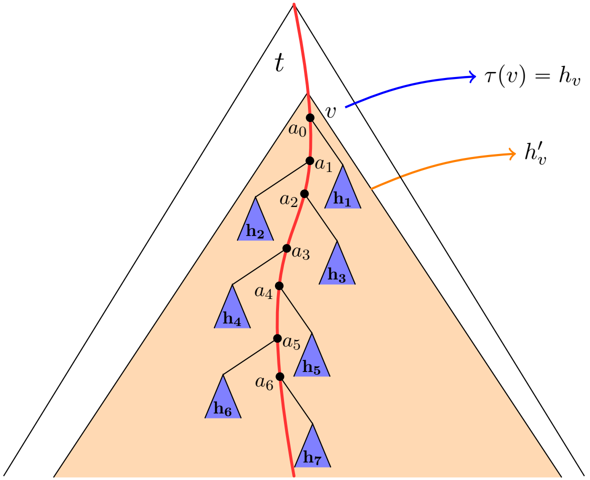

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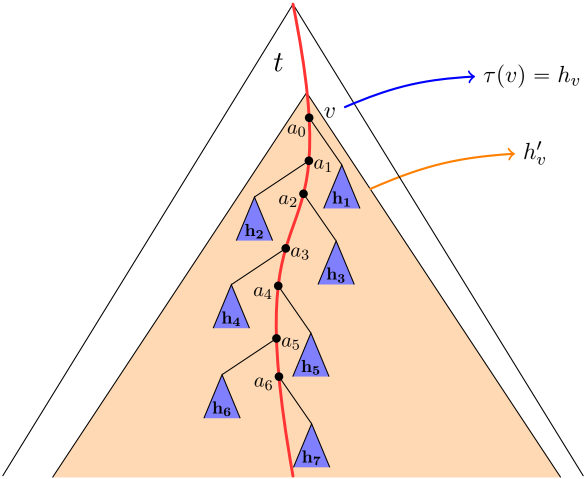

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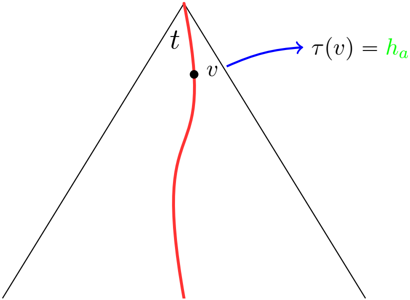

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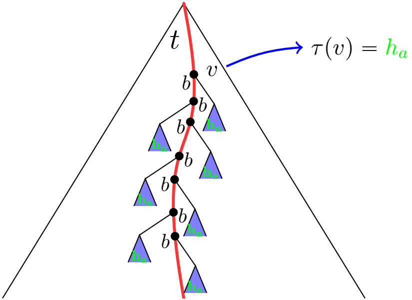

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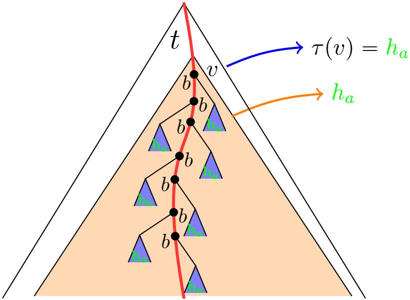

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τ is consistent!

Theorem (S. [2013])

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iff

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[it is enough to use thin algebra to check if $h_v = h'_v$]

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[no actual marking because $\alpha\colon \mathrm{Scattered} \to H$ (not $\alpha\colon \mathrm{Trees} \to H$)]

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$\rightsquigarrow$ connections with path continuous hyper-clones of Blumensath