An automata-theoretic hierarchy inside $\mathbf{\Delta}^1_2$

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replacing Damian Niwiński

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Lausanne

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Rabin's decidability of S2S

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MSO subsumes LTL, CTL*, μ-calculus, ...

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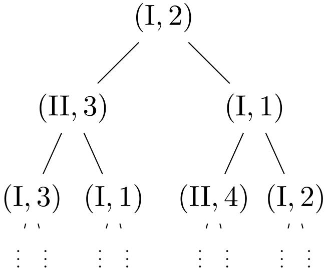

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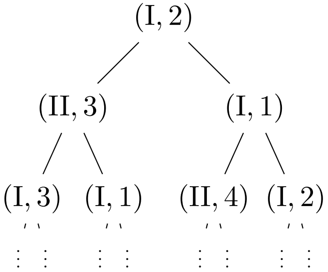

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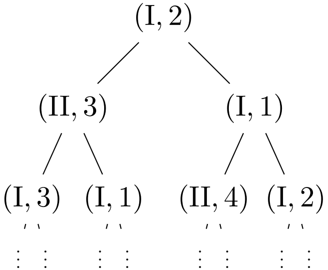

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Index hierarchy

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for every φ there effectively exists a parity automaton $\mathcal{A}$ s.t.

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Index hierarchy is the alternation-depth hierarchy for μ-calculus

Game trees

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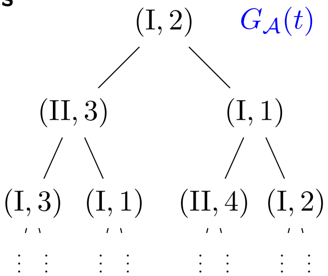

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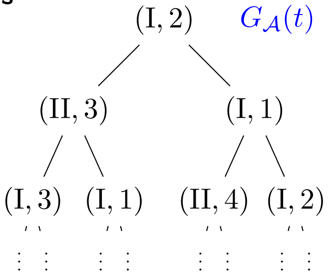

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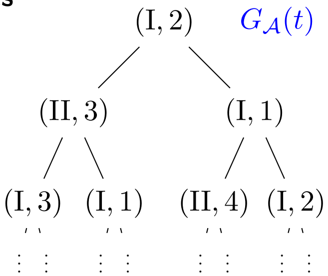

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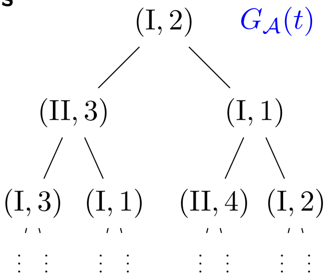

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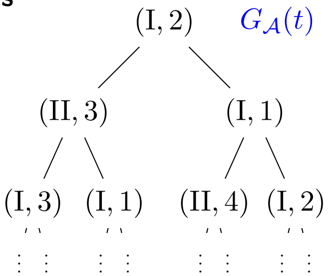

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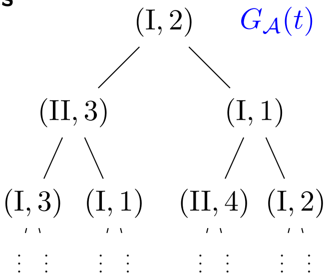

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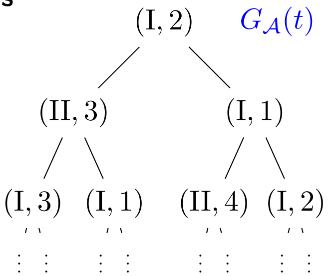

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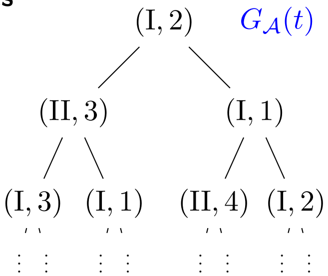

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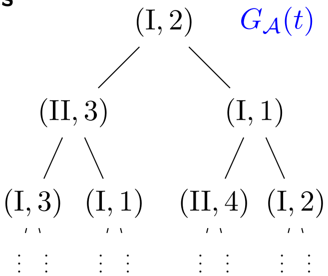

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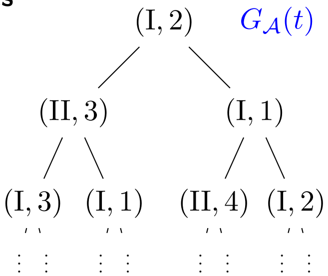

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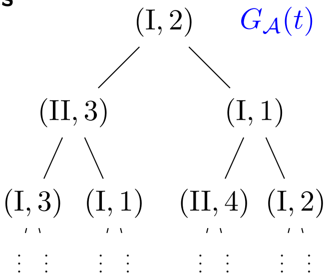

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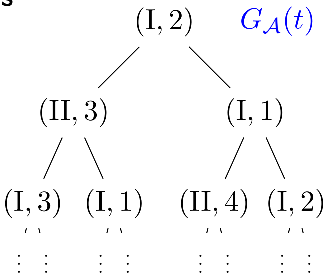

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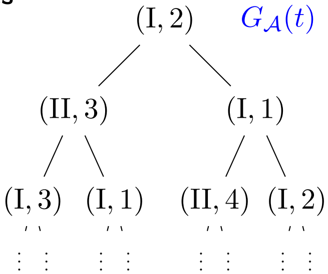

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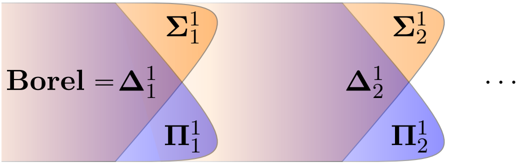

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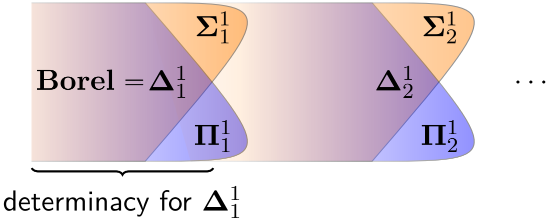

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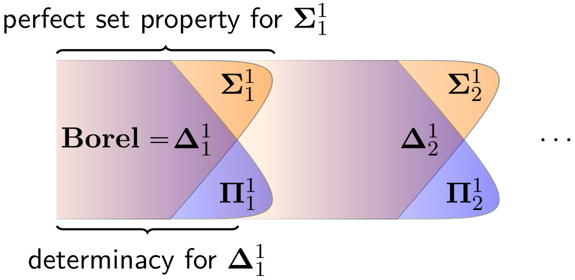

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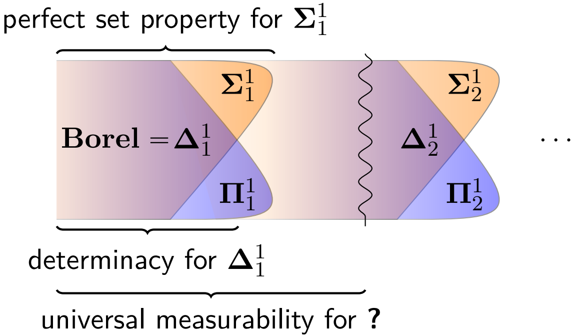

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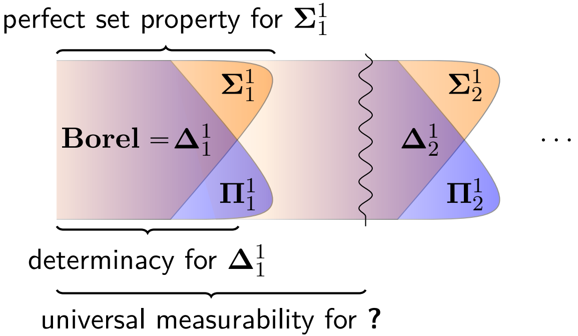

$\rightsquigarrow$ search for constructive representations of sets in $\boldsymbol{\Delta}^1_2$

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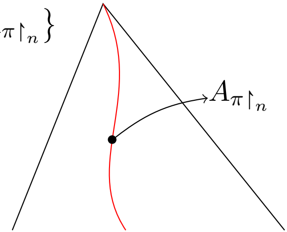

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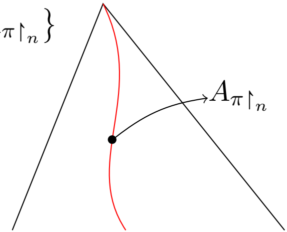

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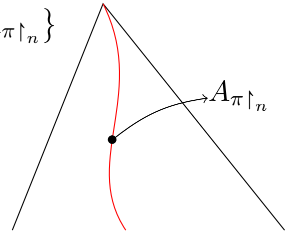

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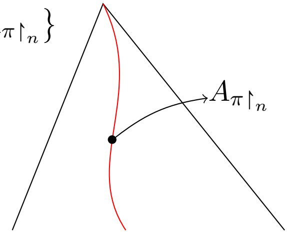

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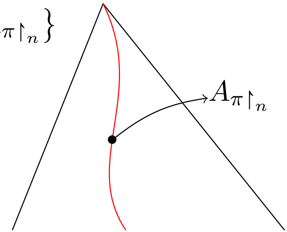

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- $\boldsymbol{\Sigma}_1^1$-inductive sets (Moschovakis 1974)

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Caution: $\forall A \subseteq X\ \exists \Omega\ \exists (A_s)_{s\in\mathbb{A}}.\quad (A_s)_s \subseteq \mathbf{\Pi}^0_1 \ \wedge\ \Omega\big((A_s)_{s\in\mathbb{A}}\big) = A$

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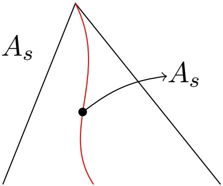

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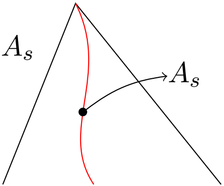

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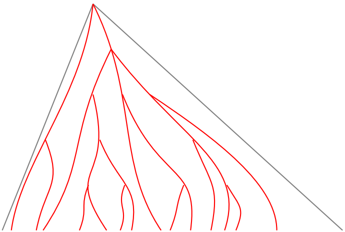

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$$\text{co-}(\mathbb{A},\mathbb{B}) = \Big(\mathbb{A},\ \underbrace{\{M \mid \forall N \in \mathbb{B}.\ N \cap M \neq \varnothing\}}_{\text{counter-strategies}}\Big)$$

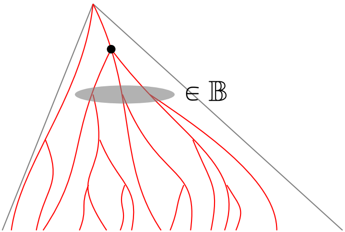
$\text{co-}\bigcup = \bigcap \qquad \text{co-}\bigcap = \bigcup \qquad X - \text{co-}\Omega\big((X - A_n)_{n\in\mathbb{A}}\big) = \Omega\big((A_n)_{n\in\mathbb{A}}\big)$

Definition (Kolmogorov [1928]) $\quad \mathcal{R}\colon \Omega \mapsto \mathcal{R}\Omega$

$$\mathcal{R}(\mathbb{A},\mathbb{B}) = \Big(\mathbb{A}^{<\omega},\ \{M \mid \epsilon \in M \ \wedge\ \forall s \in M.\ \{n \mid s\hat{\ }n \in M\} \in \mathbb{B}\}\Big)$$

Intuition:

- play ω-iterated game for $(\mathbb{A},\mathbb{B})$
- M combines original strategies



Examples

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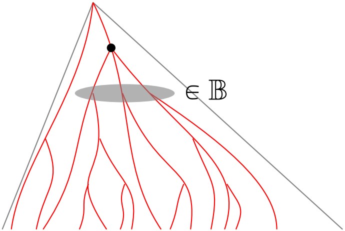

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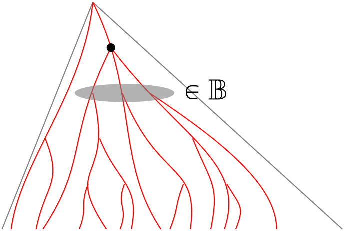

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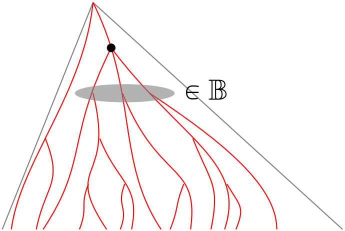

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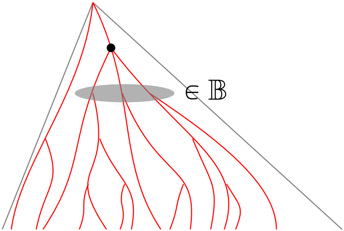

$$\mathcal{R}\big(\bigcup\big) \;=\; \Big(\omega^{<\omega},\ \big\{M \mid \epsilon \in M \ \wedge\ \forall s \in M.\ \big|\{n \mid s\hat{}n \in M\}\big| = 1\big\}\Big)$$

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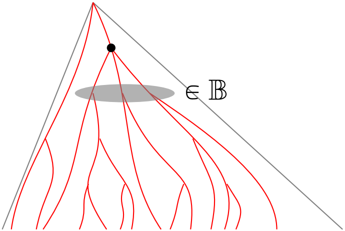

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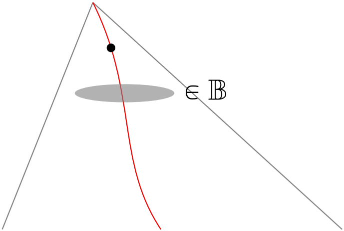

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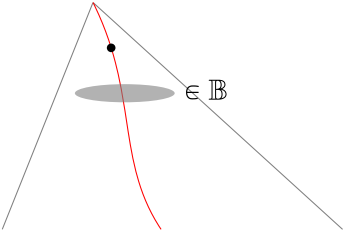

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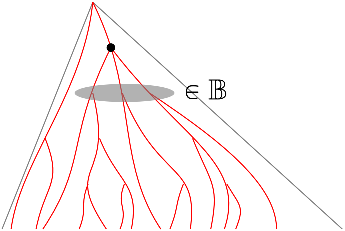

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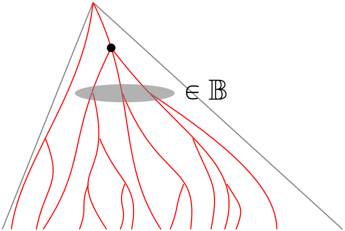

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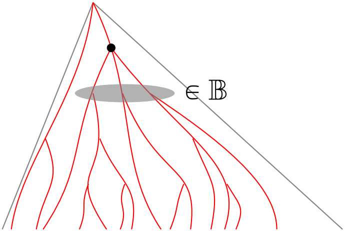

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$\mathcal{R}$-**sets** (finite levels)

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$$\bigcup \xleftarrow{\text{co-}} \bigcap$$

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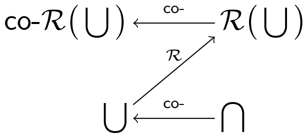

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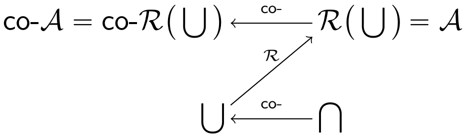

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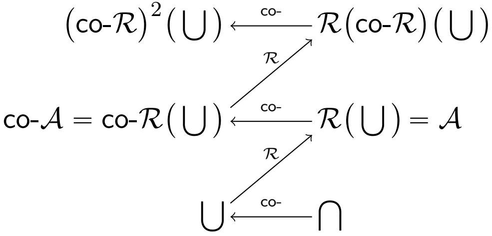

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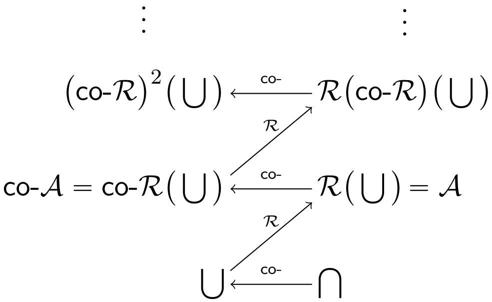

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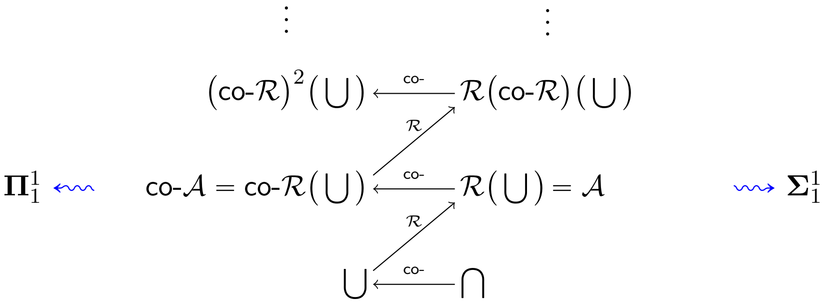

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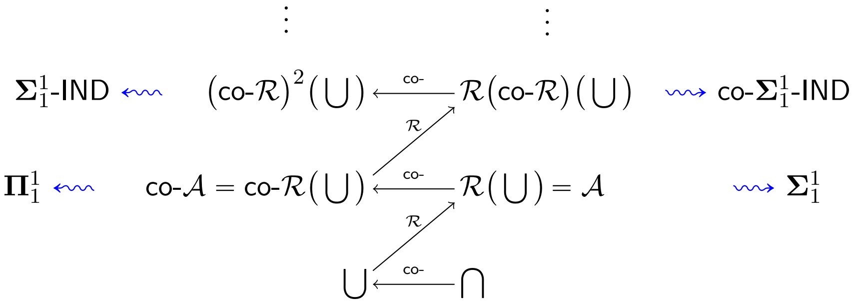

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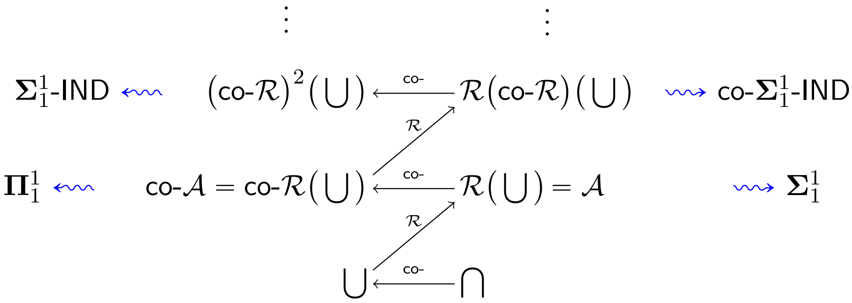

Theorem (Kolmogorov [1928], Luzin, Sierpiński [1918])
If Ω preserves measurability then co-Ω and $\mathcal{R}\Omega$ preserve measurability.

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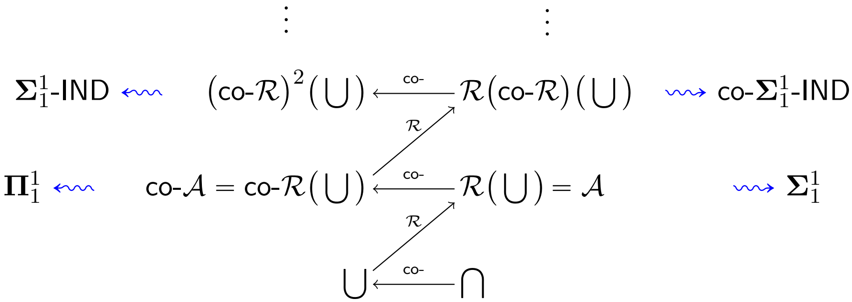

Theorem (Kolmogorov [1928], Luzin, Sierpiński [1918])

If Ω preserves measurability then co-Ω and $\mathcal{R}\Omega$ preserve measurability.

Corollary

All $\mathcal{R}$-sets are universally measurable.

Few examples. . .

Few examples...

Theorem (Saint Raymond [2006])

The set of *cofinal* trees is complete for $\left(\text{co-}\mathcal{R}\right)^2\left(\bigcup\right)(\mathbf{\Pi}^0_1)$ ($=\mathbf{\Sigma}^1_1$-IND)

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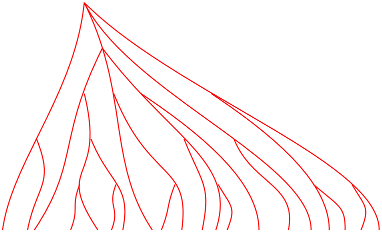

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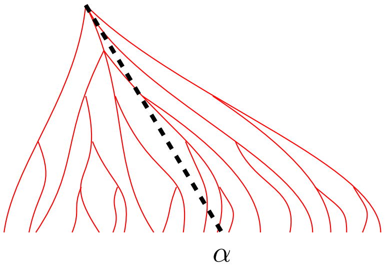

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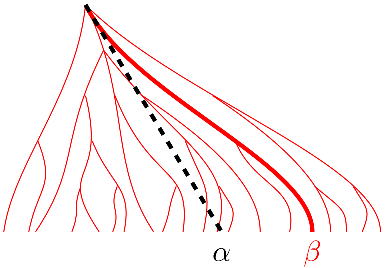

Correspondence between $\mathcal{R}$-sets and $W_{i,k}$

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Theorem (Gogacz, Michalewski, Mio, S. [2014])

The set $W_{k-1,2k-1}$ is Wadge-complete for $(\text{co-}\mathcal{R})^k(\bigcup)\left(\mathbf{\Pi}^0_1\right)$ sets.

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$$\begin{array}{rl} \vdots & \vdots \\ (\text{co-}\mathcal{R})^3(\bigcup) & \mathcal{R}(\text{co-}\mathcal{R})^2(\bigcup) \\ (\text{co-}\mathcal{R})^2(\bigcup) & \mathcal{R}(\text{co-}\mathcal{R})(\bigcup) \\ \text{co-}\mathcal{R}(\bigcup) & \mathcal{R}(\bigcup) \end{array}$$

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$$\begin{array}{llll}
\vdots & \vdots & \vdots & \vdots \\
(0,3) \subseteq & (\text{co-}\mathcal{R})^3(\bigcup) & \mathcal{R}(\text{co-}\mathcal{R})^2(\bigcup) & \supseteq (1,4) \\
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W_{0,1} \in & (0,1) \subseteq & \underbrace{\text{co-}\mathcal{R}\big(\bigcup\big) \quad \mathcal{R}\big(\bigcup\big)}_{\text{boldface hierarchy}} & & \supseteq (1,2) & \ni W_{1,2}
\end{array}$$

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boldface hierarchy

effective hierarchy

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complete sets

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The set $W_{k-1,2k-1}$ is Wadge-complete for $(\text{co-}\mathcal{R})^k(\bigcup)\left(\mathbf{\Pi}^0_1\right)$ sets.

Corollary

Every regular set of trees is universally measurable.

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Also: correspondence between parity games and $\mathcal{R}$-transform

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Proved for $(i,k)=(0,1)$ (Michalewski, Hummel, Niwiński [2009])

Proved for all odd k (Arnold, Michalewski, Niwiński [2012])

$\rightsquigarrow$ **open** for even k (except $(1,2)$)

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Does Borel rank match weak quantifier alternation

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$\rightsquigarrow$ applications to unambiguous automata

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