Descriptive set theoretic methods in automata theory

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prof. Igor Walukiewicz

Motivation:

Motivation: Theoretical Computer Science

Motivation: Theoretical Computer Science [**verification**]

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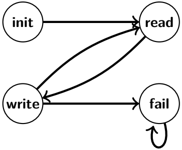
device

Motivation: Theoretical Computer Science [**verification**]

device

automaton



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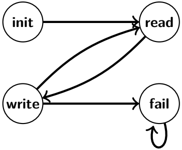
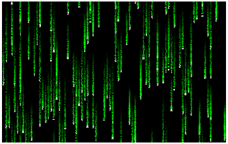


execution



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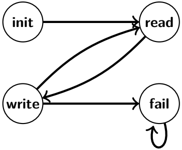

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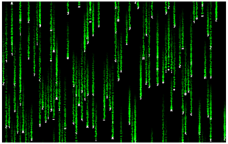
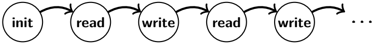


word



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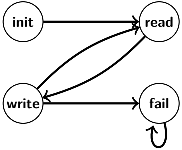

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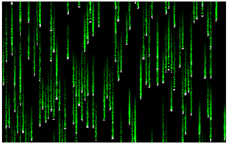

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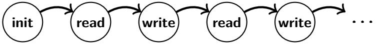

requirement

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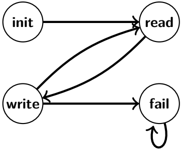

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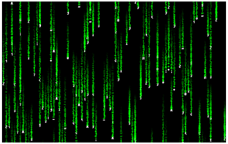

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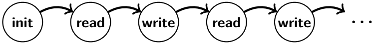


requirement

formula

$$\forall t.\ \neg \mathrm{fail}(t)$$

Logical approach

Logical approach

Structures:

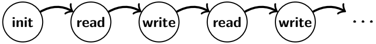
Logical approach

Structures: finite / **infinite**:

Logical approach

Structures: finite / **infinite**:

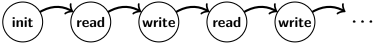
words

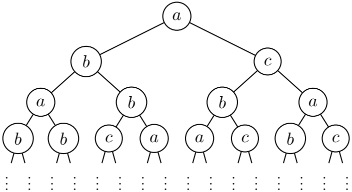


Logical approach

Structures: finite / **infinite**:

words and **trees**





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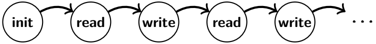

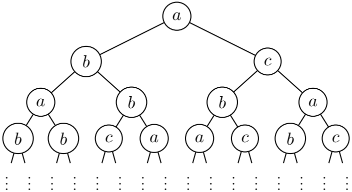

Logic:

Logical approach

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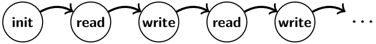

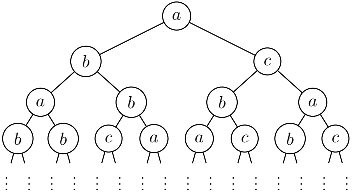

Logic: **Monadic Second-Order** (MSO) logic

Logical approach

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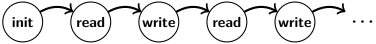

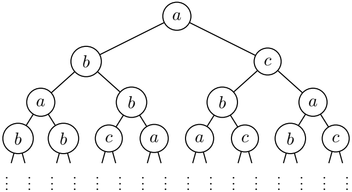

Logic: **Monadic Second-Order** (MSO) logic

- $\exists_x$, $\forall_x$ $\quad x$ — node

Logical approach

Structures: finite / **infinite**:

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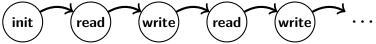

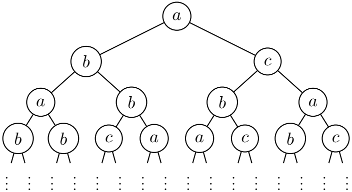

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- $\exists_X$, $\forall_X$ X — set of nodes

Logical approach

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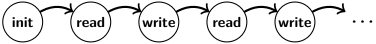

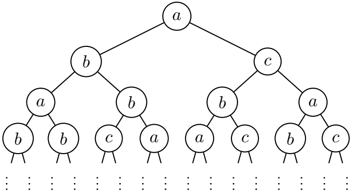

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Logical approach

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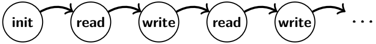

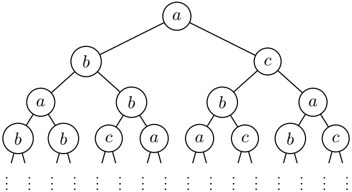

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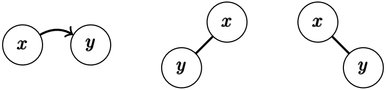
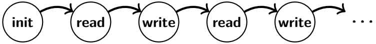
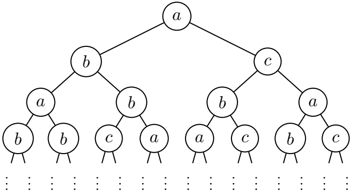


Logical approach

Structures: finite / **infinite**: **words** and **trees**





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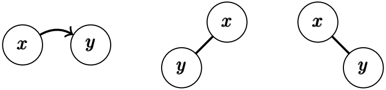

- order $\leqslant$ / $\preceq$

Logical approach

Structures: finite / **infinite**:

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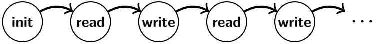

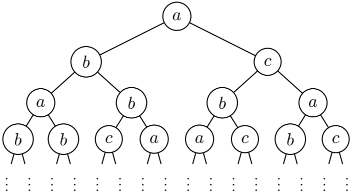

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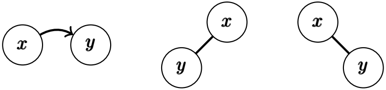

- order $\leqslant$ / $\preceq$
- label predicates: $a(x)$, $b(x)$, ...

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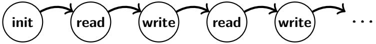
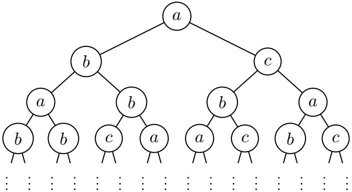
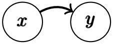
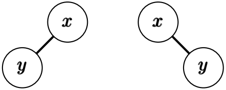




Logic: **Monadic Second-Order** (MSO) logic

- $\exists_x$, $\forall_x$ $\quad x$ — node
- $\exists_X$, $\forall_X$ $\quad X$ — set of nodes / finite set of nodes (weak MSO)
- $x \in X$, $x = y$
- successor predicates:





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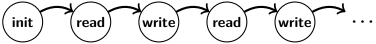
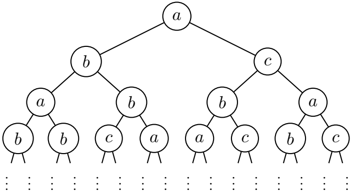
Logical approach

Structures: finite / **infinite**: **words** and **trees**





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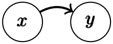

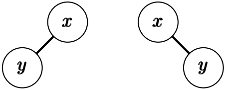

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no arithmetic !!!

Examples

Examples

Infinite words

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— safety: $\forall x.\ \neg a(x)$

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Infinite trees

— two incomparable b:

$$\exists x.\ \exists y.\ b(x) \wedge b(y) \wedge \neg\big(x \preceq y \vee y \preceq x\big)$$

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$$\exists X.\ \big(\exists x.\ x \in X\big) \wedge$$

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$$\exists X.\ \big(\exists x.\ x \in X\big) \wedge \\ \big(\forall x.\ x \in X \Rightarrow a(x)\big) \wedge$$

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MSO **subsumes**: LTL, CTL*, modal μ-calculus, ...

Decidability

Decidability

Theorem (Büchi [1962], Rabin [1969])

The Monadic Second-order logic is decidable over:

— infinite words and

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Key ingredient

Decidability

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Key ingredient

formula φ $\leftrightsquigarrow$ automaton $\mathcal{A}$

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Meta problems: given φ decide properties of $L = \{s \mid s \models \varphi\}$

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— is L empty **?**

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Theorem (Büchi [1962], Rabin [1969])

The Monadic Second-order logic is decidable over:

— infinite words and

— infinite trees.

Meta problems: given φ decide properties of $L = \overbrace{\{s \mid s \models \varphi\}}^{\text{set of words / trees satisfying } \varphi}$

— is L empty?	✓	[Büc62] / [Rab69]
— is L countable?	✓	[BKR10]
— is L First-order definable?	✓/?	[Sch65], [McNP71], [Tho79]
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Theorem (Rice [1939])

Every non-trivial property of recursively enumerable sets is undecidable.

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word labelled by $\{0, 1, \ldots, 9\}$

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[the same holds for infinite trees]

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$\rightsquigarrow$ meta-problems: is $\{t \mid t \models \varphi\}$ open, closed, etc.

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Theorem (Niwiński [1985])

There exists an MSO formula φ such that

$$\{t \mid t \models \varphi\} \text{ is non-Borel.}$$

Topology vs. **definability**

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Conjecture (Skurczyński [1993])

For every MSO-definable set of infinite trees $L = \{t \mid t \models \varphi\}$:

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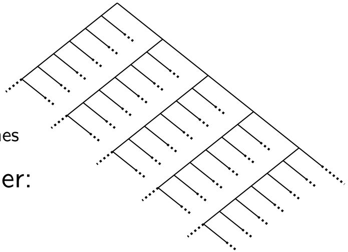

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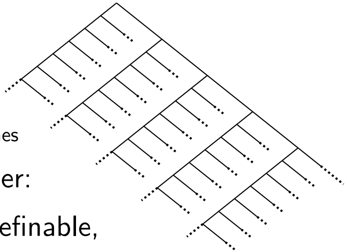

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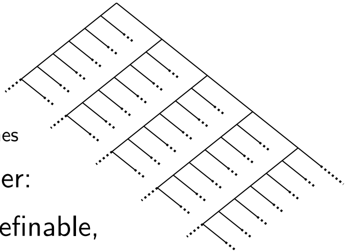

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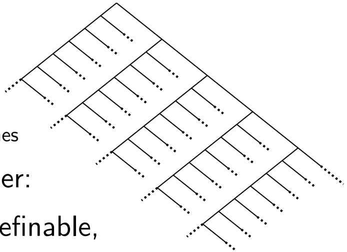

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Moreover, it is decidable which of the cases holds.

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⇝ applications to unambiguous automata

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