Measure Properties of Game Tree Languages

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MFCS 2014
Budapest

Two-player stochastic games

Two-player stochastic games

φ: winning condition ("specification": LTL, FO, ω-reg. exp., MSO, ...)

Two-player stochastic games

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$\forall$ — universal, refuter

$\exists$ — existential, prover

$\mathbf{N}$ — nature, random

Two-player stochastic games

φ: winning condition ("specification": LTL, FO, ω-reg. exp., MSO, ...)

$\forall$

$\exists$ $\pi =$

$\mathbf{N}$

Two-player stochastic games

φ: winning condition ("specification": LTL, FO, ω-reg. exp., MSO, ...)

$\forall$: a_0

$\exists$ $\pi =$

$\mathbf{N}$

Two-player stochastic games

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$\forall$: a_0

$\exists$: e_0 $\pi =$

$\mathbf{N}$

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$\forall$: a_0

$\exists$: e_0 $\quad\quad \pi =$

$\mathbf{N}$: r_0

Two-player stochastic games

φ: winning condition ("specification": LTL, FO, ω-reg. exp., MSO, ...)

$\forall$

$\exists$ $\pi = (a_0, e_0, r_0)$

$\mathbf{N}$

Two-player stochastic games

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$\forall$: a_1

$\exists$ $\pi = (a_0, e_0, r_0)$

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$\exists$: e_1 $\pi = (a_0, e_0, r_0)$

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$\forall$

$\exists$ $\pi = (a_0, e_0, r_0), (a_1, e_1, r_1)$

$\mathbf{N}$

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$\forall$: a_n

$\exists$: e_n $\qquad \pi = (a_0, e_0, r_0), (a_1, e_1, r_1), (a_2, e_2, r_2), \ldots$

$\mathbf{N}$: r_n

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φ: winning condition ("specification": LTL, FO, ω-reg. exp., MSO, ...)

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$\exists$: e_n $\qquad \pi = \underbrace{(a_0, e_0, r_0), (a_1, e_1, r_1), (a_2, e_2, r_2), \ldots}_{\text{linear time play}}$

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$\mathbf{N}$: r_n

$\exists$ wins if $\pi \models \varphi$

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Determinacy:

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Determinacy:

$$\sup_{\sigma_\exists} \inf_{\sigma_\forall} \mathbb{P}_{\vec{r}}\Big(\pi \models \varphi\Big) = \inf_{\sigma_\forall} \sup_{\sigma_\exists} \mathbb{P}_{\vec{r}}\Big(\pi \models \varphi\Big)$$

Determinacy (linear time)

$$\sup_{\sigma_\exists} \inf_{\sigma_\forall} \quad \mathbb{P}_{\vec{r}} \Big(\pi \ \models \ \varphi \Big) \ = \ \inf_{\sigma_\forall} \sup_{\sigma_\exists} \quad \mathbb{P}_{\vec{r}} \Big(\pi \ \models \ \varphi \Big)$$

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1. Measurability (linear time)

Is $\Big\{\pi : \ \pi \models \varphi\Big\}$ measurable?

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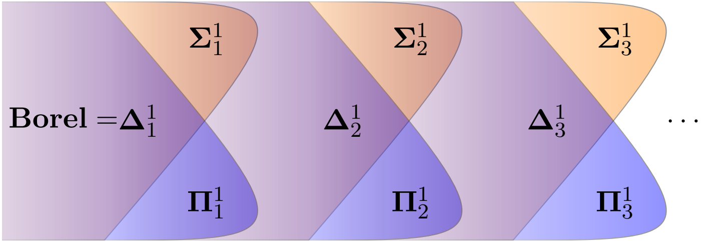

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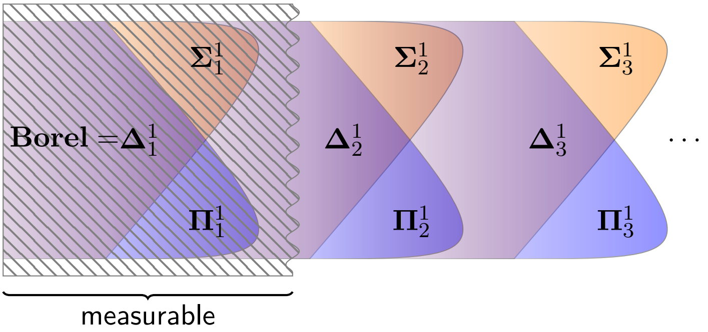

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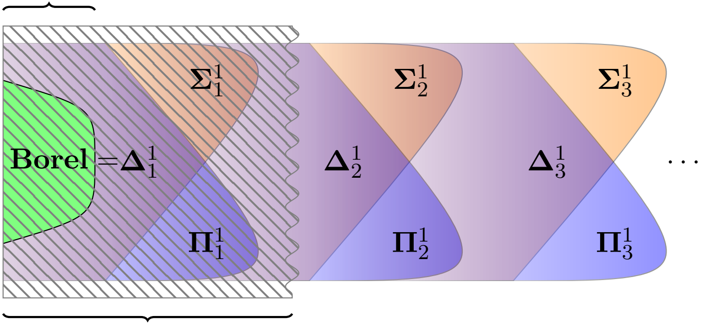
1. Measurability (linear time)

MSO over ω-words

Is $\{\pi : \pi \models \varphi\}$ measurable?



measurable

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1. **Measurability (linear time)** ✓

2. **Approximability (linear time)**

$$\forall_{\epsilon>0} \exists_{\sigma_\exists} \quad \mathrm{val}(\sigma_\exists) \geq \inf_{\sigma_\forall} \mathrm{val}(\sigma_\forall) - \epsilon$$

Determinacy (linear time)

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e.g. (Chatterjee Jurdziński Henzinger [2004])

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e.g. (Chatterjee Jurdziński Henzinger [2004])

[holds in general for Borel games (Martin [1998])]

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Two-player stochastic tree games (Mio [2012])

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[game semantics for probabilitic μ-calculus with independent product]

Two-player stochastic tree games (Mio [2012])

$\forall$

$\exists$

$\mathbf{N}$

Two-player stochastic tree games (Mio [2012])

$\forall$

$\exists$ $t =$

$\mathbf{N}$

Two-player stochastic tree games (Mio [2012])

$\forall$: a_ϵ

$\exists$ $t =$

$\mathbf{N}$

Two-player stochastic <u>tree</u> games (Mio [2012])

$\forall$: a_ϵ

$\exists$: e_ϵ $t =$

$\mathbf{N}$

Two-player stochastic tree games (Mio [2012])

$\forall$: a_ϵ

$\exists$: e_ϵ $\qquad t =$

$\mathbf{N}$: r_ϵ

Two-player stochastic tree games (Mio [2012])

$\forall$ $(a_\epsilon, e_\epsilon, r_\epsilon)$

$\exists$ $t =$

$\mathbf{N}$

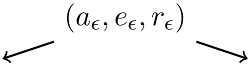
Two-player stochastic tree games (Mio [2012])

∀

∃

$\mathbf{N}$

$t =$



Two-player stochastic tree games (Mio [2012])

$\forall$: a_d

$\exists$: e_d

$\mathbf{N}$: r_d

$$t = \begin{array}{ccc} & (a_\epsilon, e_\epsilon, r_\epsilon) & \\ \swarrow & & \searrow \\ (a_{\mathsf{L}}, e_{\mathsf{L}}, r_{\mathsf{L}}) & & (a_{\mathsf{R}}, e_{\mathsf{R}}, r_{\mathsf{R}}) \end{array}$$

Two-player stochastic tree games (Mio [2012])

$\forall$

$\exists$

$\mathbf{N}$

$$t = \quad (a_\epsilon, e_\epsilon, r_\epsilon)$$

$$(a_{\mathsf{L}}, e_{\mathsf{L}}, r_{\mathsf{L}}) \qquad (a_{\mathsf{R}}, e_{\mathsf{R}}, r_{\mathsf{R}})$$

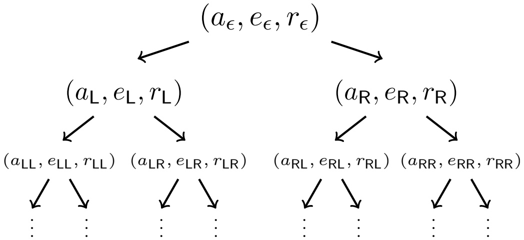
Two-player stochastic tree games (Mio [2012])

$\forall$: a_u

$\exists$: e_u

$\mathbf{N}$: r_u

$t =$



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$\forall$: a_u

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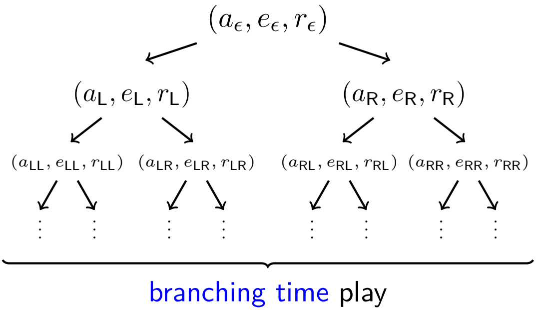

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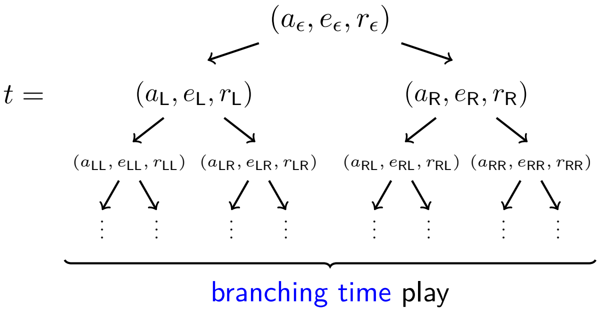

$\exists$ wins if $t \models \varphi$

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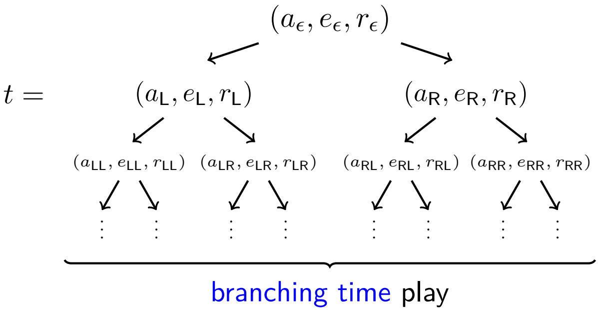


$\exists$ wins if $t \models \varphi$

Determinacy:

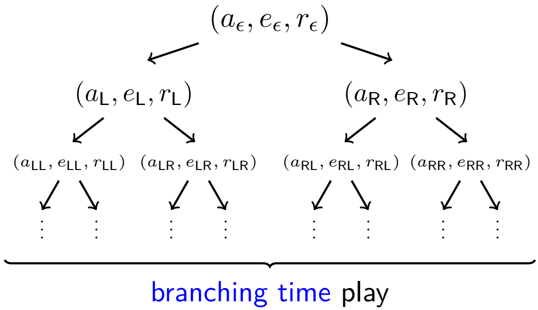
Two-player stochastic tree games (Mio [2012])

$\forall$: a_u

$\exists$: e_u

$\mathbf{N}$: r_u

$t =$



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Determinacy (branching time)

$$\sup_{\sigma_{\exists}} \inf_{\sigma_{\forall}} \quad \mathbb{P}_{r}\Big(t \models \varphi\Big) = \inf_{\sigma_{\forall}} \sup_{\sigma_{\exists}} \quad \mathbb{P}_{r}\Big(t \models \varphi\Big)$$

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1. Measurability (branching time)

Determinacy (branching time)

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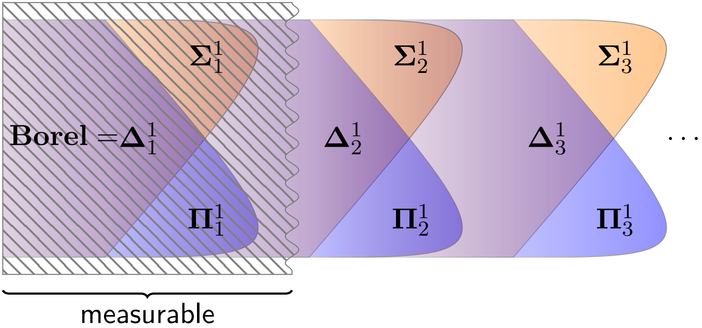

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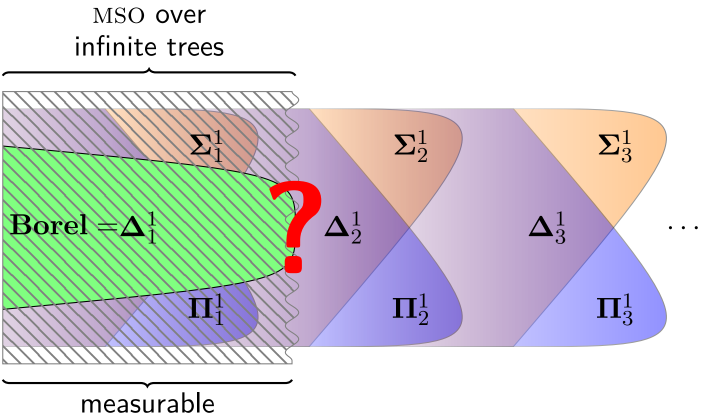

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$$\sup_{\sigma_\exists} \inf_{\sigma_\forall} \quad \underset{r}{\mathbb{P}}\Big(t \models \varphi\Big) = \inf_{\sigma_\forall} \sup_{\sigma_\exists} \quad \underset{r}{\mathbb{P}}\Big(t \models \varphi\Big)$$

1. Measurability (branching time) ?

Determinacy (branching time)

$$\sup_{\sigma_\exists} \inf_{\sigma_\forall} \underbrace{\mathbb{P}_r\Big(t \models \varphi\Big)} = \inf_{\sigma_\forall} \sup_{\sigma_\exists} \underbrace{\mathbb{P}_r\Big(t \models \varphi\Big)}$$

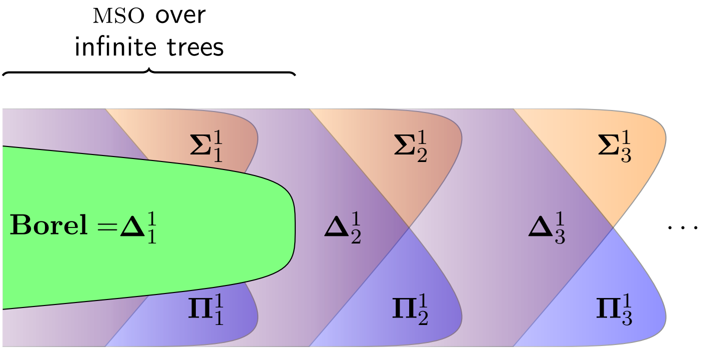
1. Measurability (branching time) ?

M. Mio [2012] : assume $\mathbf{MA}_{\aleph_1}$

Determinacy (branching time)

$$\sup_{\sigma_\exists} \inf_{\sigma_\forall} \underset{r}{\mathbb{P}}\left(t \models \varphi\right) = \inf_{\sigma_\forall} \sup_{\sigma_\exists} \underset{r}{\mathbb{P}}\left(t \models \varphi\right)$$

1. Measurability (branching time) (assuming $\mathbf{MA}_{\aleph_1}$)



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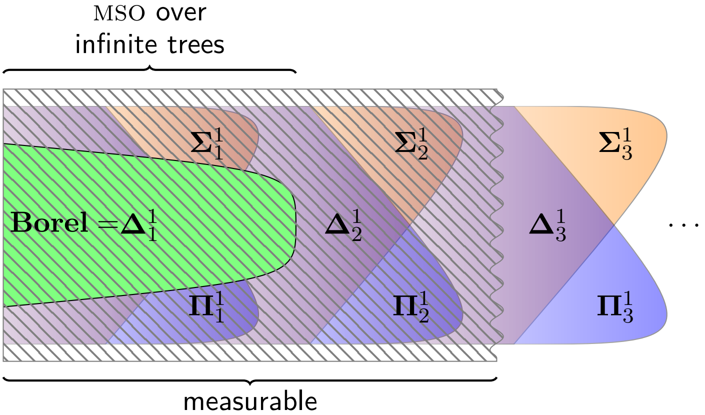

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1. Measurability (branching time) (assuming $\mathbf{MA}_{\aleph_1}$) ✓ } (Mio [2012])

Determinacy (branching time)

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1. **Measurability (branching time)** (assuming $\mathbf{MA}_{\aleph_1}$) ✓
2. **Approximability (branching time)** (assuming $\mathbf{MA}_{\aleph_1}$) ✓

} (Mio [2012])

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1. **Measurability (branching time)** (assuming $\mathbf{MA}_{\aleph_1}$) ✔
2. **Approximability (branching time)** (assuming $\mathbf{MA}_{\aleph_1}$) ✔

} (Mio [2012])

Open problem: what happens without $\mathbf{MA}_{\aleph_1}$?

Subcase: only random player $\mathbf{N}$

Subcase: only random player N

Flip a coin in every node of a tree over $\{a, b\}$:

Subcase: only random player $\mathbb{N}$

Flip a coin in every node of a tree over $\{a, b\}$:

$$\mathbb{P}\left(t(u) = a\right) = \frac{1}{2}$$

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$\mathbb{P}\left(\{\text{“}a\text{ in the root”}\}\right)$

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$\inf_a \stackrel{\text{def}}{=}$ "set of branches containing infinitely many a"

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$\inf_a \overset{\text{def}}{=}$ “set of branches containing infinitely many a”

$\mathbb{P}\left(\{\text{“}\inf_a \text{ is non-empty”}\}\right)$

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$\mathbb{P}\Big(\{\text{"}\inf_a \text{ is non-empty"}\}\Big) \;=\; 1$

$\mathbb{P}\Big(\{\text{"}\inf_a \text{ is uncountable"}\}\Big)$

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Flip a coin in every node of a tree over $\{a, b\}$:

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$\inf_a \stackrel{\text{def}}{=}$ “set of branches containing infinitely many a”

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$\mathbb{P}\Big(\{\text{“}\inf_a \text{ is uncountable”}\}\Big) = \textbf{?}$ (is it measurable?)

Our main result

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Theorem

Every MSO*-definable set of infinite trees is measurable.*

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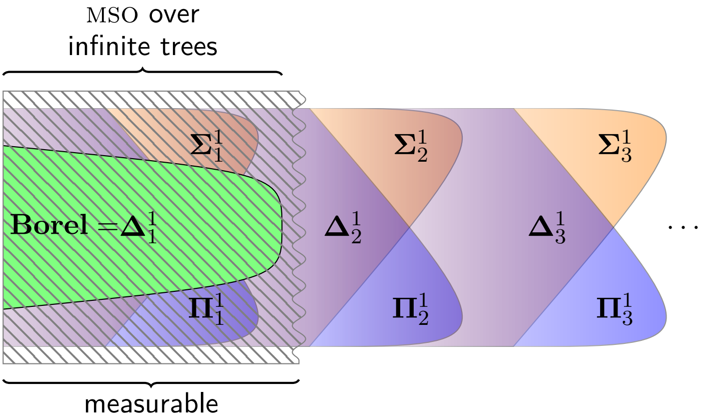

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Main ingredient: $\mathcal{R}$-transformation (Kolmogorov [1928])

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- languages $\mathbf{W}_{i,k}$

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- $\mathcal{R}$-transformation

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Theorem
Every MSO*-definable set of infinite trees is measurable.*

Main ingredient: $\mathcal{R}$-transformation (Kolmogorov [1928])

Plan:

- languages $\mathbf{W}_{i,k}$

- $\mathcal{R}$-transformation

- equivalence via Matryoshka games

Languages $\mathbf{W}_{i,k}$ (Walukiewicz [1996])

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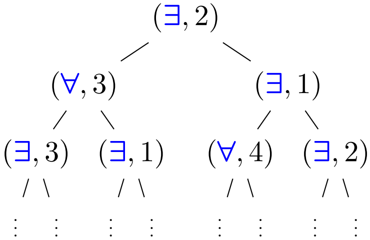
tree t over $A_{i,k}$



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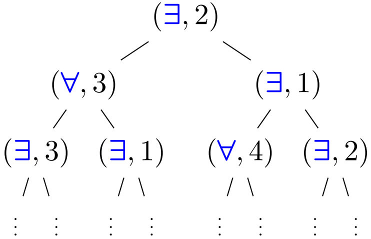
tree t over $A_{i,k}$ $\rightsquigarrow$ parity game $\mathcal{G}_t$



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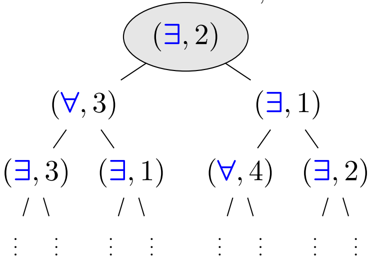

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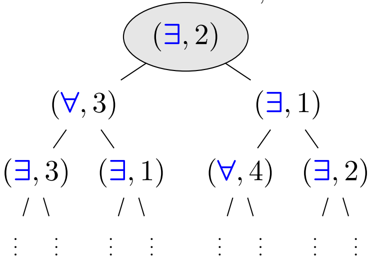

$\exists$: d_0 $\quad$ (= L)

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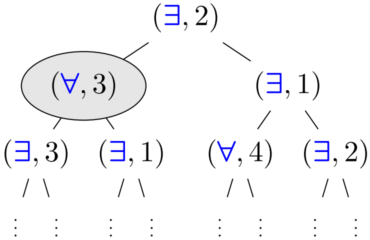

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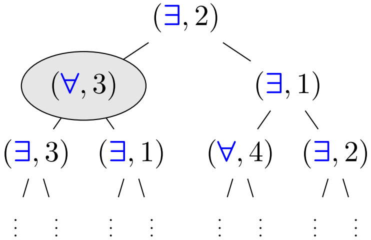

$\forall \; : d_1 \quad (= \mathsf{R})$

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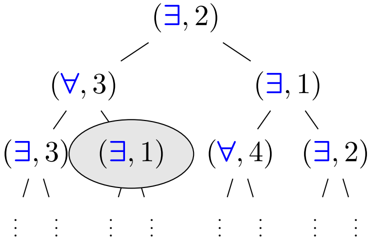

$\pi = 2, 3$

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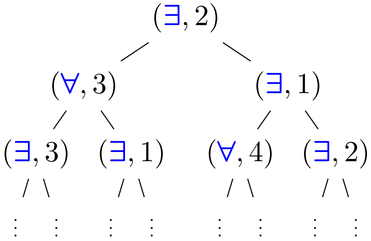

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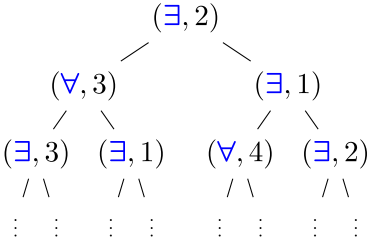

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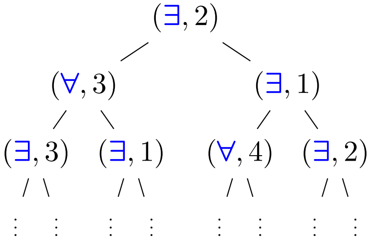

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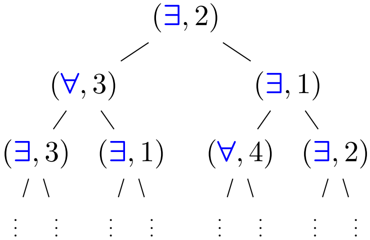

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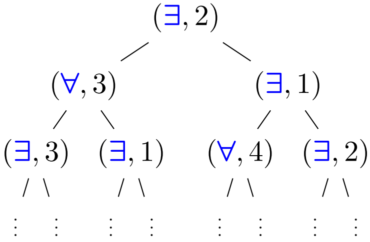

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for every φ there exists (i, k) such that:

$\{t : t \models \varphi\}$ is topologically simpler than $\mathbf{W}_{i,k}$

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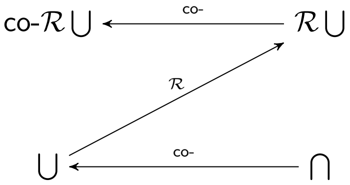

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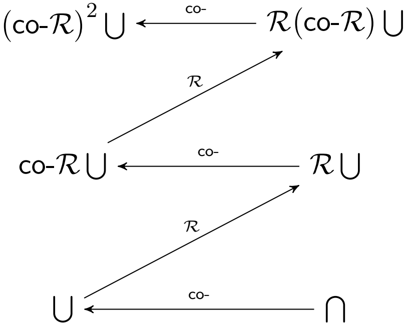

$$\mathcal{R}(\text{co-}\mathcal{R}) \bigcup \sim \underbrace{\text{co-}\left[\cup(\cup(\ldots))\right]\left(\text{co-}\left[\cup(\cup(\ldots))\right]\left(\ldots\right)\right)}_{\omega^2\text{-composition}}$$

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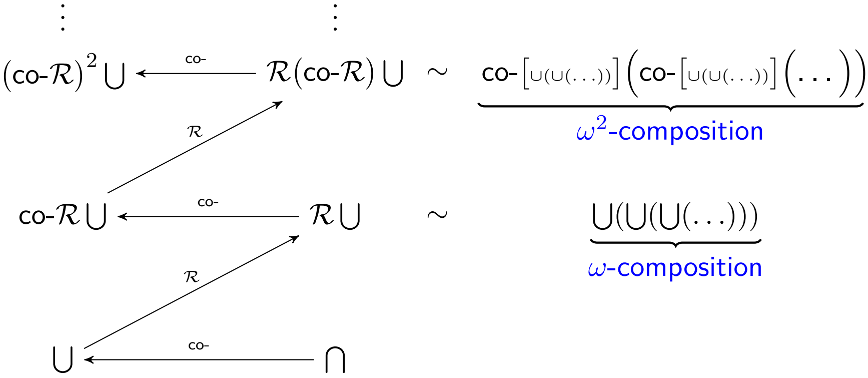

$\mathcal{R}$-sets (Kolmogorov [1928] ... Barua [1986])

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k-level of the $\mathcal{R}$-hierarchy:

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Proof.

Γ preserves measurability

$$\Downarrow$$

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■

Matryoshka games

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Parity games with exits

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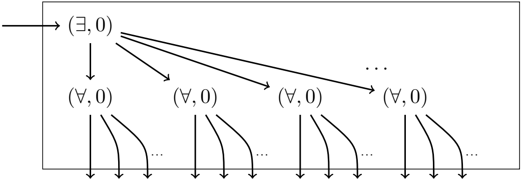

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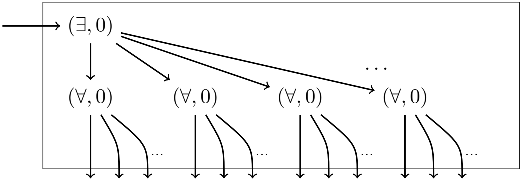

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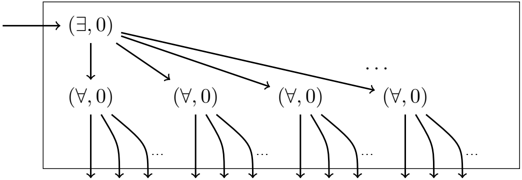
Matryoshka games



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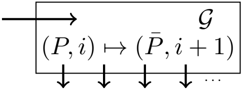
$\mathcal{G} \mapsto$ co-$\mathcal{G}$



"dual game"

Matryoshka games

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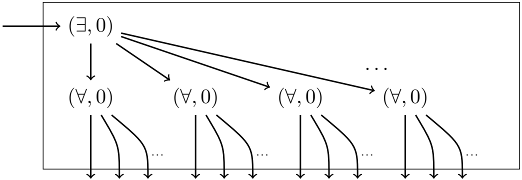

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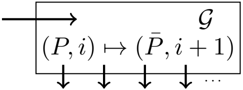

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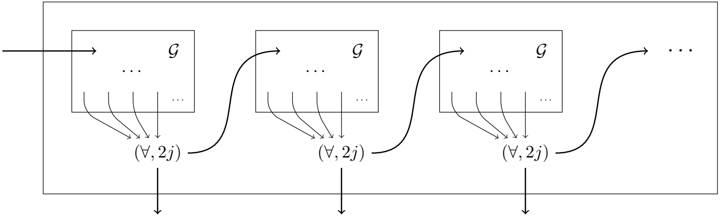

Matryoshka games

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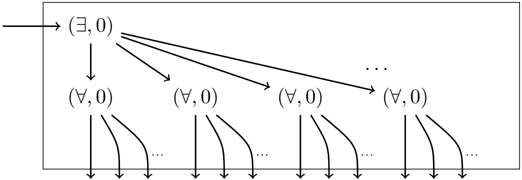

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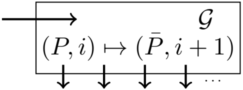

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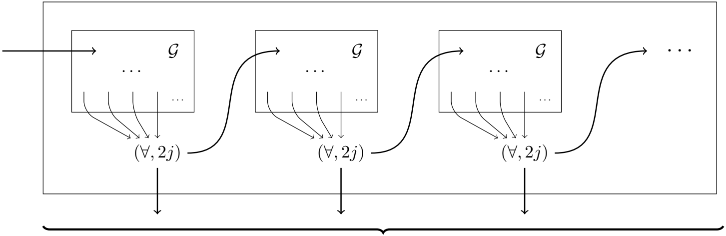

ω-composition

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In extended version:

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In extended version:

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