Deciding the index hierarchies of game automata

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University of Warsaw

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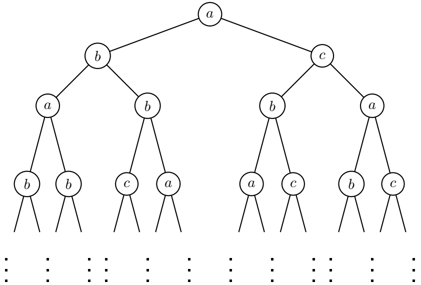
New Orleans

Infinite trees are very rich



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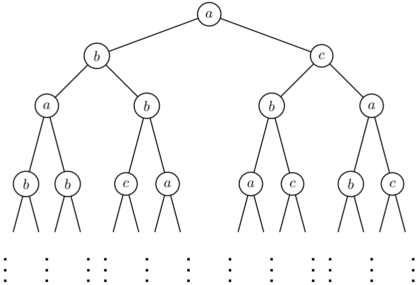

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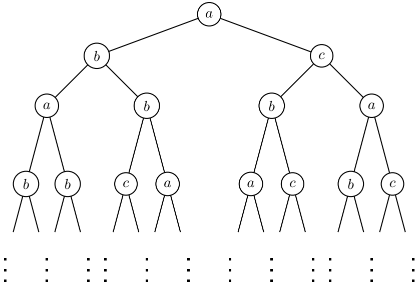

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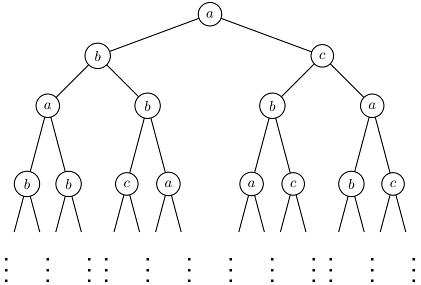

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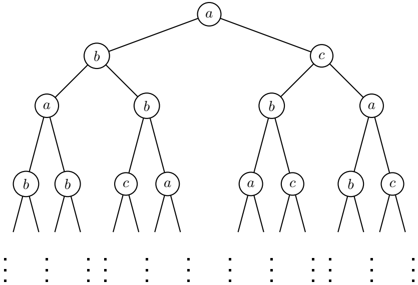

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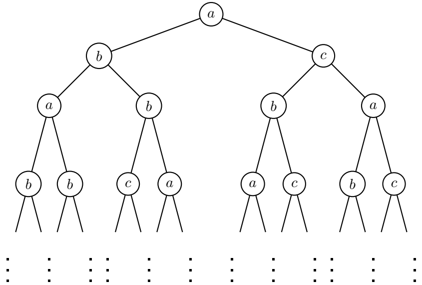

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$\rightsquigarrow$ Application in verification and model-checking

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- FO, CTL*, . . .
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Rabin-Mostowski index problem

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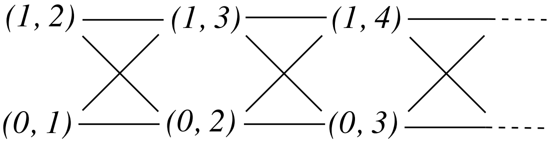

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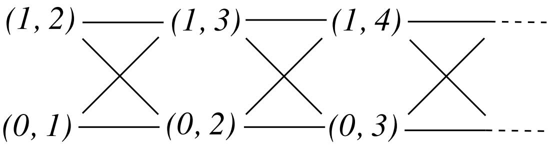

$\rightsquigarrow$ alternation of fixpoints in modal μ-calculus

Infinite trees are difficult

Only few effective characterizations:

- Boolean combinations of open sets [Bojańczyk, Place '12]
- nondetereministic $(0, 1)$-automata [Colcombet, Löding]
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Approach: solve problems for "easier" subclasses

- all hierarchies decidable for deterministic automata [Niwiński, Walukiewicz '03, '05; M '05, '06, '08]
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This work:
Rabin-Mostowski index problem for game automata

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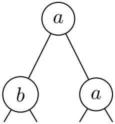

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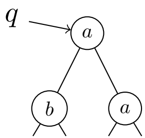

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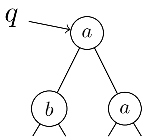

Semantics via games: two opponents ∃ and ∀

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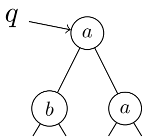

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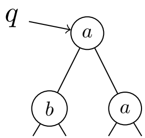

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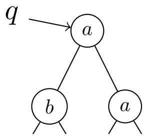

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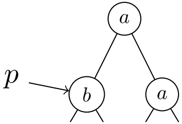

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A tree t is accepted if $\exists$ has a winning strategy

Game automata

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$\rightsquigarrow$ every node of a tree can be reached in exactly one state

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- recognize languages arbitrarily high in both hierarchies.

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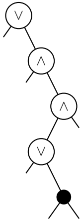

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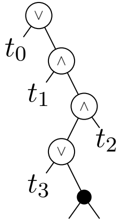

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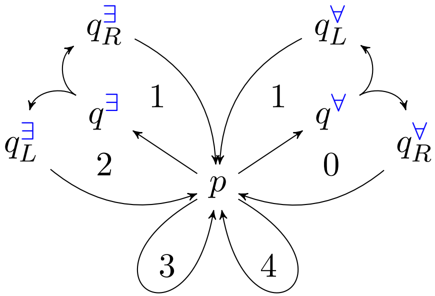

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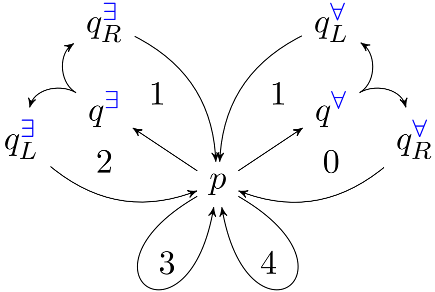
Using tools of topology



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Inductively construct a *simple* automaton recognising $\mathrm{L}(\mathcal{A})$.

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Game automata are as manageable as deterministic ones.