Complexity collapse for unambiguous languages

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Paris

Existential quantifier $\rightsquigarrow$ projection

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$$S = \left\{ x \ : \ \exists_y \ R(x, y) \right\}$$

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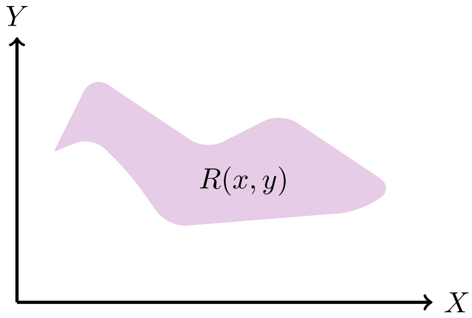

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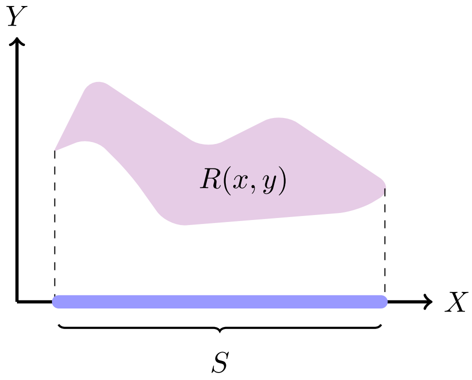
Existential quantifier ⇝ projection



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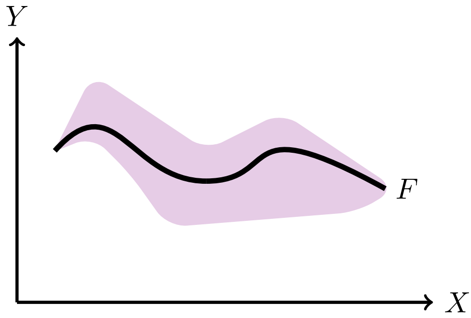

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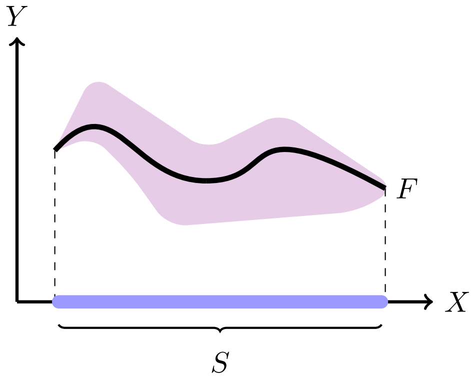

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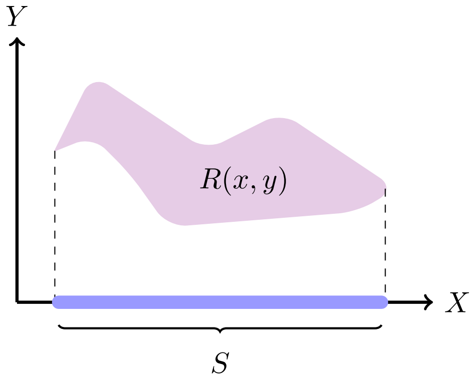

Theorem (~~Lebesgue~~, Souslin)

*Projection of a Borel set may **not** be Borel.*

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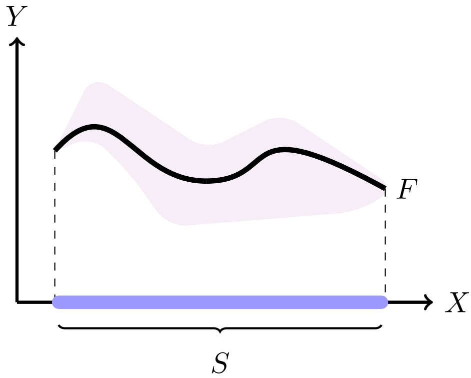

Theorem (~~Lebesgue~~, Souslin)

Projection of a Borel set may **not** *be Borel.*

Theorem (Lusin, Souslin)

Projection of an uniformized Borel set **is** *Borel.*

Automata

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Nondeterministic

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Nondeterministic parity

Nondeterministic parity tree automata

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MSO	$\equiv$	parity
existential MSO	$\equiv$	Büchi
weak MSO	$\equiv$	Büchi $\cap$ (Büchi)^c (= weak)

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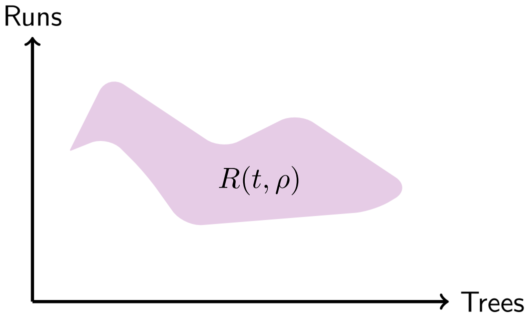

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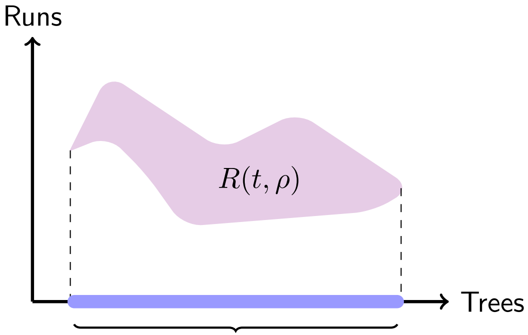

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Projection $\rightsquigarrow$ nondeterminism

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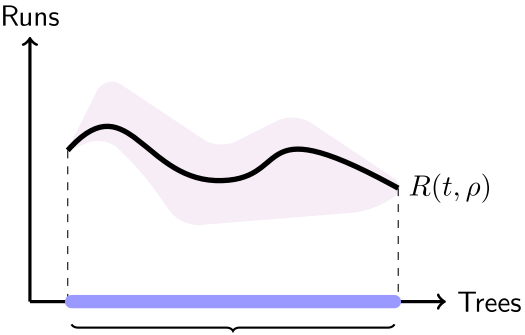
$\mathcal{A}$ is unambiguous if $\forall_t\ \exists_\rho^{\leq 1}\ \rho$ is accepting



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Lower / upper bounds for descriptive complexity of unambiguous languages.

(Partial answer by Hummel [2012], [2013]:

There are unambiguous languages above $\mathbf{\Pi}^1_1$.)

Unambiguous Büchi is Borel

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Theorem (Finkel, Simmonet [2009])

If $\mathcal{A}$ *is unambiguous* **and** *Büchi then* $\mathrm{L}(\mathcal{A})$ *is Borel.*

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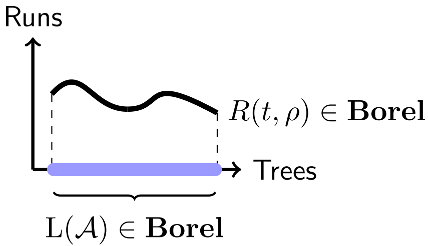
Proof.



■

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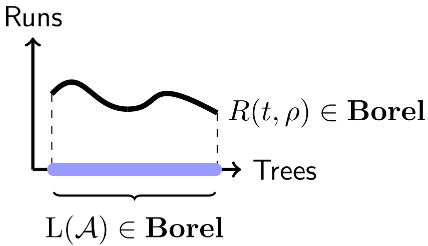

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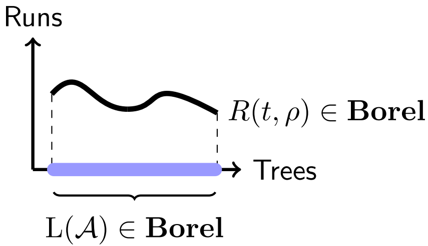

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But what if:

Conjecture (Skurczyński [1993])

If a $\mathrm{L}(\mathcal{A})$ is Borel then $\mathrm{L}(\mathcal{A})$ is weak MSO-definable.

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There exists a language L that is:

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- recognised by a Büchi (but not unambiguous) automaton,
- non-Borel.

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Hopefully a step towards upper bounds for unambiguous languages.