Equational theories of profinite structures

Michał Skrzypczak

University of Warsaw

August 23'rd 2011

http://www.mimuw.edu.pl/~mskrzypczak/docs/

Note

Things that I show are nothing remarkably *new*. This is rather a point of view than a new piece of theory.

Note

Things that I show are nothing remarkably *new*. This is rather a point of view than a new piece of theory.

Profinite structures

Add *virtual* objects to our world to make it more *complete* (e.g. compact).

Note

Things that I show are nothing remarkably *new*. This is rather a point of view than a new piece of theory.

Profinite structures

Add *virtual* objects to our world to make it more *complete* (e.g. compact).

Equational theories

What properties of languages can be expressed by (some) equations?

Definition

A framework is a pair $\langle \Phi, \mathbb{W} \rangle$ such that:

- Φ is a countable set of *recognisers* $\varphi \in \Phi$,
- $\mathbb{W}$ is a countable set of *objects* $w \in \mathbb{W}$,
- a recogniser $\varphi \in \Phi$ is a function $\varphi\colon \mathbb{W} \to K_\varphi$ to a finite set K_φ.

Definition

A framework is a pair $\langle \Phi, \mathbb{W} \rangle$ such that:

- Φ is a countable set of *recognisers* $\varphi \in \Phi$,
- $\mathbb{W}$ is a countable set of *objects* $w \in \mathbb{W}$,
- a recogniser $\varphi \in \Phi$ is a function $\varphi \colon \mathbb{W} \to K_\varphi$ to a finite set K_φ.

Running example

Let $\mathbb{W} = A^*$ be a set of all finite words and let Φ be the set of all homomorphisms into finite monoids: for every finite monoid M and any homomorphism $\varphi \colon A^* \to M$ let $\varphi \in \Phi$.

Definition

A set $L \subseteq \mathbb{W}$ is *recognisable* if there exists a recogniser $\varphi \in \Phi$ and a set $V \subseteq K_\varphi$ such that

$$L = \varphi^{-1}(V).$$

Definition

A set $L \subseteq \mathbb{W}$ is *recognisable* if there exists a recogniser $\varphi \in \Phi$ and a set $V \subseteq K_\varphi$ such that

$$L = \varphi^{-1}(V).$$

Assumptions

Additionally we assume:

a) Each object $w \in \mathbb{W}$ is totally described by some recogniser (that is $\{w\}$ is recognisable).

b) Recognisable sets are closed under intersections.

Examples

- Let $\mathbb{W}$ be the set of all finite models of a fixed relational signature Σ.
- Let Φ be the set of all first order formulas over Σ.
- A formula φ is a function $\varphi\colon \mathbb{W} \to \{\bot, \top\}$.

Examples

- Let $\mathbb{W}$ be the set of all finite models of a fixed relational signature Σ.
- Let Φ be the set of all first order formulas over Σ.
- A formula φ is a function $\varphi \colon \mathbb{W} \to \{\bot, \top\}$.

- Let $\mathbb{W}$ be the set of all finite labelled trees over a finite alphabet A.
- Let Φ be the set of all morphisms into finite tree algebras.

Examples

- Let $\mathbb{W}$ be the set of all finite models of a fixed relational signature Σ.
- Let Φ be the set of all first order formulas over Σ.
- A formula φ is a function $\varphi\colon \mathbb{W} \to \{\bot, \top\}$.

- Let $\mathbb{W}$ be the set of all finite labelled trees over a finite alphabet A.
- Let Φ be the set of all morphisms into finite tree algebras.

- Let $\mathbb{W}$ be the set of all finite words A^*.
- Let Φ be the set of all total (halting) Turing machines.
- Every total Turing machine M can be treated as a function $M\colon \mathbb{W} \to \{\text{accept}, \text{reject}\}$.

Definition

Let

$$X = \prod_{\varphi \in \Phi} K_\varphi .$$

X is a compact topological space. Let

$$w \in \mathbb{W} \;\mapsto\; \mu(w) = (\varphi_1(w), \varphi_2(w), \varphi_3(w), \ldots)$$

Since μ is 1-1 we can identify w with $\mu(w)$ and write $\mathbb{W} \subseteq X$. Let

$$\widehat{\mathbb{W}} = cl\,(\mathbb{W}) \subseteq X.$$

Intuitions

- For an object $w \in \mathbb{W}$ the image

$$\mu(w) = (\varphi_1(w), \varphi_2(w), \ldots) \in X$$

is a list of values of all recognisers on w. It *describes* all important properties of w.

Intuitions

- For an object $w \in \mathbb{W}$ the image

$$\mu(w) = (\varphi_1(w), \varphi_2(w), \ldots) \in X$$

  is a list of values of all recognisers on w. It *describes* all important properties of w.
- The image $\mu(\mathbb{W}) \subseteq X$ is a set of all possible (*realisable*) properties of objects.

Intuitions

- For an object $w \in \mathbb{W}$ the image

$$\mu(w) = (\varphi_1(w), \varphi_2(w), \ldots) \in X$$

  is a list of values of all recognisers on w. It *describes* all important properties of w.
- The image $\mu(\mathbb{W}) \subseteq X$ is a set of all possible (*realisable*) properties of objects.
- A *virtual* object $w' \in \widehat{\mathbb{W}} \setminus \mathbb{W}$ is just a list of its properties $(v_1, v_2, \ldots)$ that are finitely realisable by real objects.

Examples

- Let $\langle \Phi, \mathbb{W} \rangle$ be the framework of directed finite graphs and first order formulas.

Examples

- Let $\langle \Phi, \mathbb{W} \rangle$ be the framework of directed finite graphs and first order formulas.
- Let w_n be an empty n-vertex graph.

Examples

- Let $\langle \Phi, \mathbb{W} \rangle$ be the framework of directed finite graphs and first order formulas.
- Let w_n be an empty n-vertex graph.
- There is an *virtual* graph $w_\infty \in \widehat{\mathbb{W}}$ such that $w_n \to w_\infty$.

Examples

- Let $\langle \Phi, \mathbb{W} \rangle$ be the framework of directed finite graphs and first order formulas.
- Let w_n be an empty n-vertex graph.
- There is an *virtual* graph $w_\infty \in \widehat{\mathbb{W}}$ such that $w_n \to w_\infty$.
- Observe that w_∞ is not so virtual — it can be seen as infinite empty graph.

Examples

- Let $\langle \Phi, \mathbb{W} \rangle$ be the framework of directed finite graphs and first order formulas.
- Let w_n be an empty n-vertex graph.
- There is an *virtual* graph $w_\infty \in \widehat{\mathbb{W}}$ such that $w_n \to w_\infty$.
- Observe that w_∞ is not so virtual — it can be seen as infinite empty graph.
- This is not a coincidence — Compactness Theorem.

Properties

Fact

All recognisers naturally extend to $\widehat{\mathbb{W}}$ *as projections.*

Properties

Fact

All recognisers naturally extend to $\widehat{\mathbb{W}}$ *as projections.*

Fact

Profinitely recognisable sets are exactly closures of normal recognisable sets.

Properties

Fact

All recognisers naturally extend to $\widehat{\mathbb{W}}$ as projections.

Fact

Profinitely recognisable sets are exactly closures of normal recognisable sets.

Fact

A set $L \subseteq \widehat{\mathbb{W}}$ is recognisable iff it is closed and open.

Equations

Definition

For $u, v \in \widehat{\mathbb{W}}$ we say that a recognisable language $L \subseteq \widehat{\mathbb{W}}$ satisfies equation $u \to v$ iff.

$$u \in L \ \Rightarrow v \in L.$$

Equations

Definition

For $u, v \in \widehat{\mathbb{W}}$ we say that a recognisable language $L \subseteq \widehat{\mathbb{W}}$ satisfies equation $u \to v$ iff.

$$u \in L \ \Rightarrow v \in L.$$

A set of equations $\mathcal{E}$ defines the family of all recognisable languages that satisfy all those equations.

Equations

Definition

For $u, v \in \widehat{\mathbb{W}}$ we say that a recognisable language $L \subseteq \widehat{\mathbb{W}}$ satisfies equation $u \to v$ iff.

$$u \in L \ \Rightarrow v \in L.$$

A set of equations $\mathcal{E}$ defines the family of all recognisable languages that satisfy all those equations.

Theorem

A family of recognisable languages $\mathcal{L}$ is definable by some equations $\mathcal{E}$ iff $\mathcal{L}$ is a lattice.

Theorem

Theorem

A family of recognisable languages $\mathcal{L}$ is definable by some equations $\mathcal{E}$ iff $\mathcal{L}$ is a lattice.

Theorem

Theorem

A family of recognisable languages $\mathcal{L}$ is definable by some equations $\mathcal{E}$ iff $\mathcal{L}$ is a lattice.

Lemma

If $I \subseteq \mathcal{L}$ and $K = \bigcup I$ is recognisable then $K \in \mathcal{L}$.
If $I \subseteq \mathcal{L}$ and $K = \bigcap I$ is recognisable then $K \in \mathcal{L}$.

Theorem

Theorem

A family of recognisable languages $\mathcal{L}$ is definable by some equations $\mathcal{E}$ iff $\mathcal{L}$ is a lattice.

Lemma

If $I \subseteq \mathcal{L}$ and $K = \bigcup I$ is recognisable then $K \in \mathcal{L}$.
If $I \subseteq \mathcal{L}$ and $K = \bigcap I$ is recognisable then $K \in \mathcal{L}$.

Sketch of the proof ($\Leftarrow$)

Take any lattice $\mathcal{L}$ and let $\mathcal{E}$ contain all equations satisfied by $\mathcal{L}$.
Take any language L satisfying all $\mathcal{E}$ and show that $L \in \mathcal{L}$.
Use above Lemma to approximate L *from inside* and *from outside*.
If it fails, then there is an equation $u \to v$ not satisfied by L — a contradiction.

References

- Mai Gehrke, Serge Grigorieff, and Jean-Éric Pin. Duality and equational theory of regular languages. In ICALP, pages 246–257, 2008.
- Mai Gehrke, Serge Grigorieff, and Jean-Éric Pin. A topological approach to recognition. In Automata, Languages and Programming, volume 6199 of LNCS, pages 151–162. Springer Berlin / Heidelberg, 2010.