Beyond sets with atoms
definability in first-order logic

2017 Category Theory in Physics, Mathematics and Philosophy

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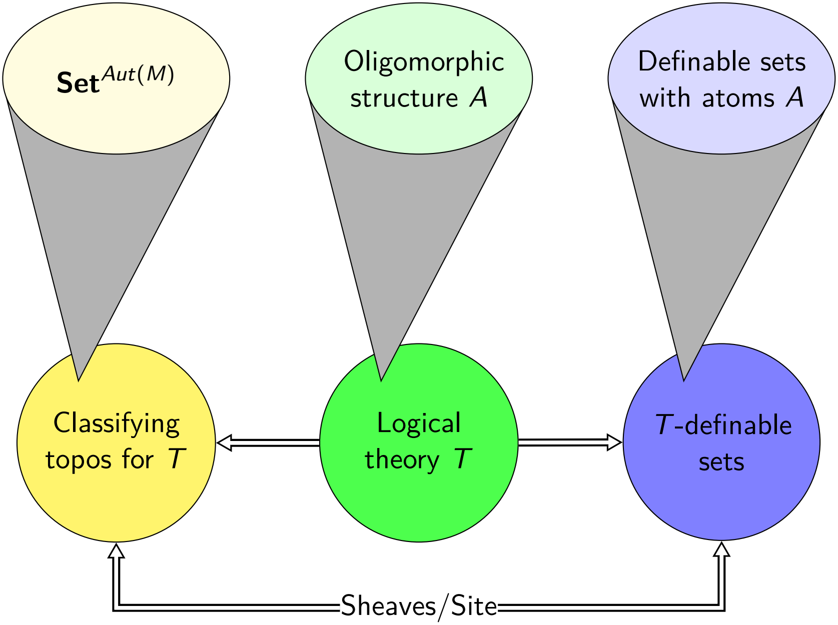
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Definable sets with atoms A
Classifying topos for T
Logical theory T
T-definable sets
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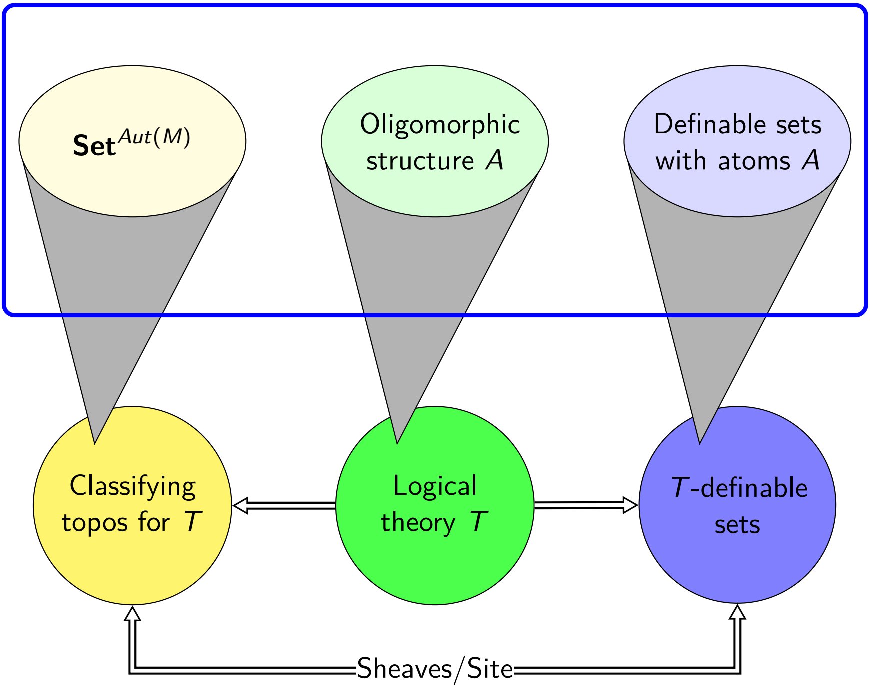

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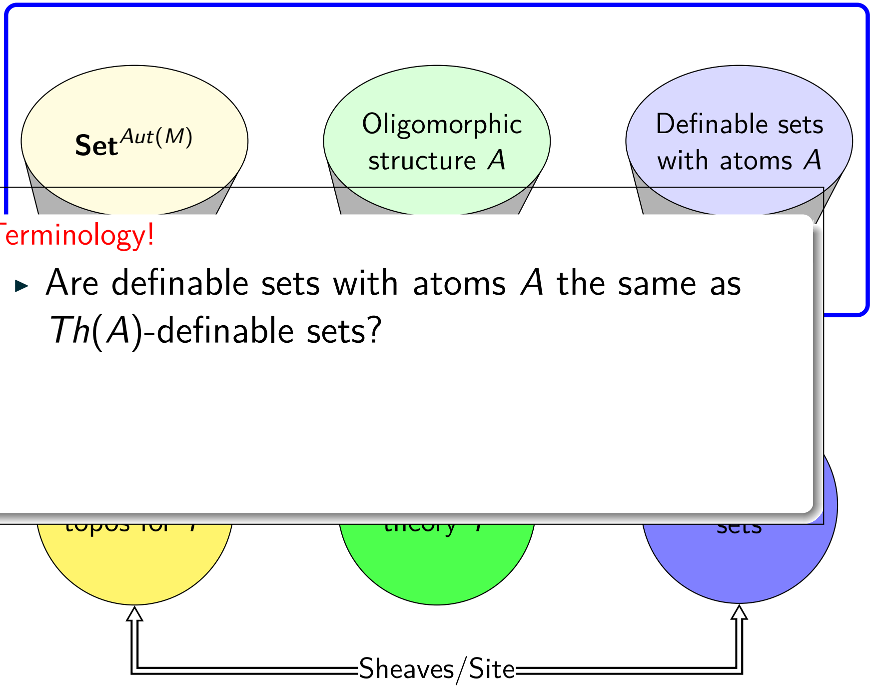

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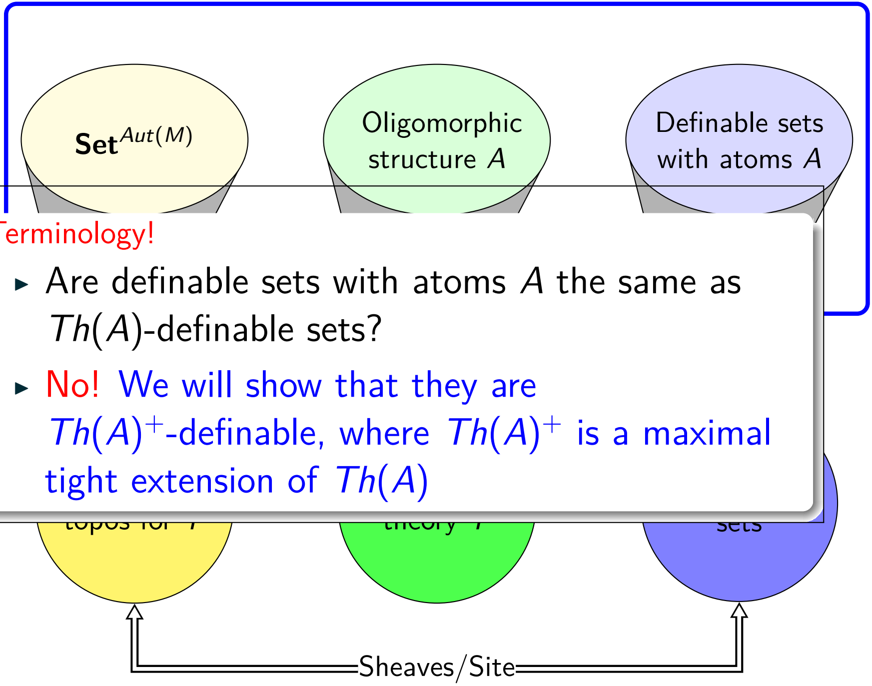
- Are definable sets with atoms A the same as $Th(A)$-definable sets?



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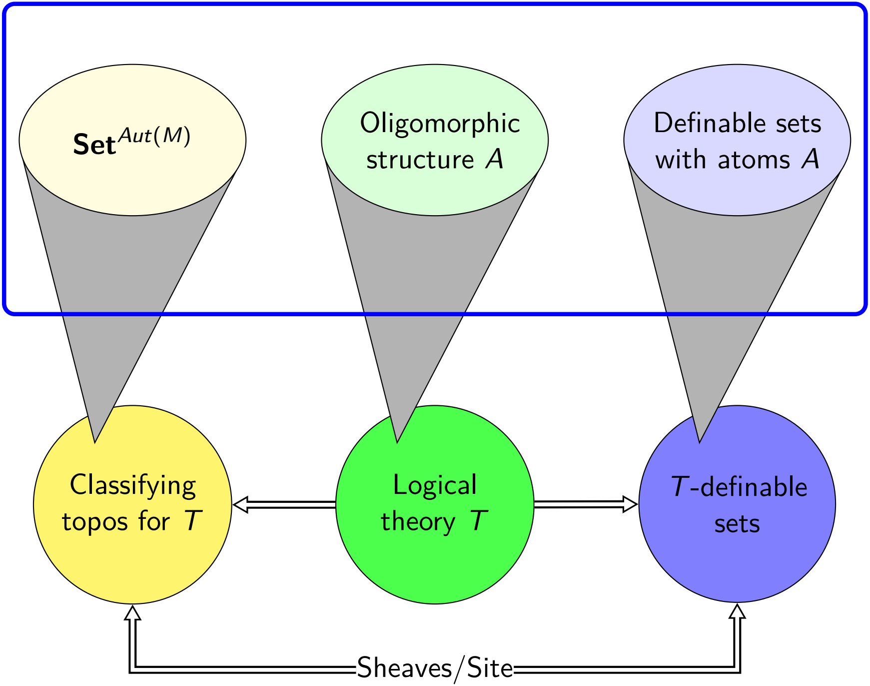
- Are definable sets with atoms A the same as $Th(A)$-definable sets?
- No! We will show that they are $Th(A)^+$-definable, where $Th(A)^+$ is a maximal tight extension of $Th(A)$



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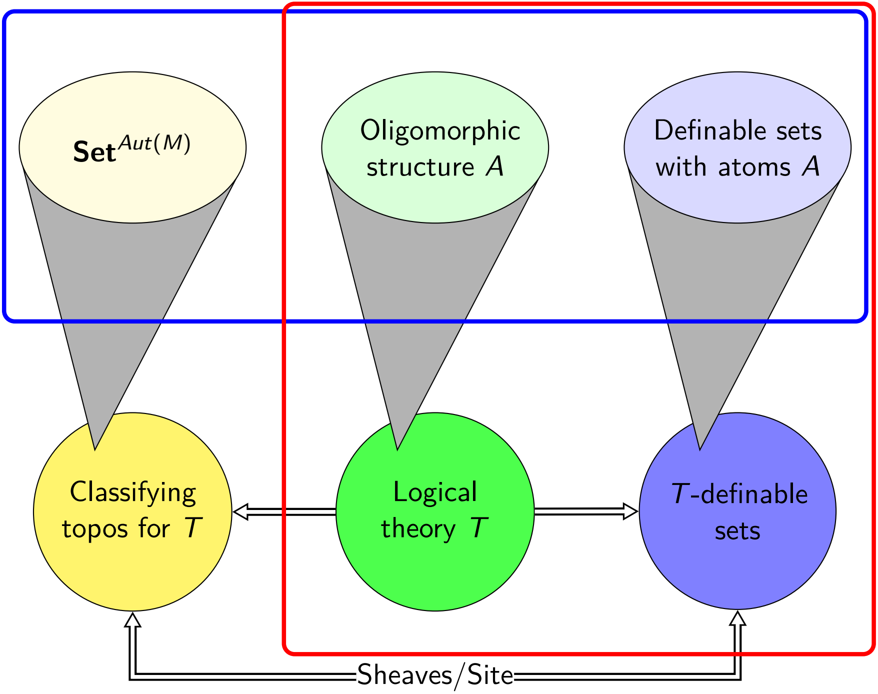
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Definable sets
Logic

- A first-order signature Σ consists of:
  - a collection of sorts $(A_i)_{i \in I}$ indexed by a set I
  - a collection of function symbols $f\colon A_{i_1} \times A_{i_2} \times \cdots \times A_{i_k} \to A_j$
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- The First-Order logic (over Σ)
  - formulas are build from terms of Σ together with:
  - relation symbols from Σ with equality $=$, and:
  - logical connectives: $\exists$, $\forall$, $\bot$, $\top$, $\vee$, $\wedge$

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- Infinitary First-Order logic allows infinite disjunctions $\bigvee$
- A theory is a set of formulas closed under logical consequence (in a given logic)

Definable sets
Example: the theory of infinite objects

- The theory of infinite objects *Eq*:
  - The empty signature with a single sort A
  - For every n, the axiom saying that there are at least n elements:

$$\exists_{x_1}\exists_{x_2}\cdots\exists_{x_n} x_1 \neq x_2 \wedge \cdots \wedge x_i \neq x_j \cdots \wedge x_{n-1} \neq x_n$$

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- All *Eq*-models of cardinality $\aleph_0$ are isomorphic (i.e. *Eq* is ω-categorical)

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Example: the theory of algebraically closed fields

- The theory of algebraically closed fields ACF:
  - A single sort C
  - Constants: $0\colon C$, $1\colon C$
  - Functions: $+\colon C \times C \to C$, $*\colon C \times C \to C$
  - Axioms expressing that $\langle C, 0, 1, +, * \rangle$ is a field
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- Example: complex numbers.
- There is an ACF of characteristic 2, i.e. $1 + 1 = 0$

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Sets

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- If $\phi(x_1, x_2, \ldots, x_n)$ has free variables $x_1, x_2, \ldots, x_n$ of sorts $A_1, A_2, \ldots, A_n$, then a set defined by it will be denoted by:

$$\{\langle x_1, x_2, \ldots, x_n\rangle \in A_1 \times A_2 \times \cdots \times A_n \colon \phi(x_1, x_2, \ldots, x_n)\}$$

or more compactly by: $\{\overline{x} \in \prod_i A_i \colon \phi(\overline{x})\}$

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- There is one interesting set in context $A \times A$:

$$\{\langle x, y\rangle \in A \times A \colon x \neq y\}$$

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- Is the last set non-empty? No — because it has no members when interpreted in the complex numbers!
- Is it empty? No — because it has members when interpreted in ACF of characteristic 2!

Definable sets
Complete theories

- Fix a first-order theory T
- Assume that T is complete and has a model M
- T-definable sets may be thought of as genuine subsets of M^K:

$$\{\langle x_1, x_2, \ldots, x_n\rangle \in A_1^M \times A_2^M \times \cdots \times A_n^M \colon \phi(x_1, x_2, \ldots, x_n)\}$$

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- Example:
  - ACF0 — the theory of algebraically closed fields of characteristic 0 is complete and has a model $\mathbb{C}$ (complex numbers)
  - ACF0-definable sets are solutions to polynomial equations (with definable coefficients), e.g.:

$$\begin{aligned} \{\langle x, y\rangle \in \mathbb{C}^2 \colon x^2 + y^2 = 1 \wedge y = 2x\} &= \{-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\} \\ \{x \in \mathbb{C} \colon x + x = 0 \wedge x \neq 0\} &= \emptyset \end{aligned}$$

Definable sets
Functions

- Fix a first-order theory T
- Definable function $f\colon \{\overline{x} \in \prod A_i \colon \phi(\overline{x})\} \to \{\overline{y} \in \prod B_i \colon \psi(\overline{y})\}$ is the equivalence class of a subformula $f(\overline{x}, \overline{y})$ of $\{\langle \overline{x}, \overline{y} \rangle \in \prod A_i \times \prod B_i \colon \phi(\overline{x}) \wedge \psi(\overline{y})\}$ that is functional:
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Syntactic category

T-definable sets together with T-definable functions form a category $\mathbb{T}$ — the syntactic category of T. Moreover, for definable A, B, f, the following sets are definable:

- $A \times B, A \cup B, f[A], f^{-1}[A]$

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$\mathbb{T}$ is a Heyting category — i.e. it has finite limits, stable unions and quantifiers. But it may lack effective quotients!

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Imaginary elements

Elements of X/R are called imaginary elements of T. A theory T (uniformly) eliminates imaginaries if for every T-definable equivalence relation R on X, the quotient set X/R together with the canonical injection $e\colon X \to X/R$ are T-definable.

Definable sets
Elimination of imaginaries

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- $\mathbb{T}^{+}$ is a pretopos

Theorem:

If A is single-sorted and countable then $Th(A)^{+} = Th(A)^{eq}$

Definable sets
Definable sets subsume sets with atoms

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Theorem:

Sets with atoms A are exactly $Th(A)^{eq}$-definable sets.

Proof (sketch):

- ($\Rightarrow$) If $\phi(\overline{x}, \overline{y})$ is an equivalence formula, then $\{\langle \overline{x}, \{\overline{y} \colon \phi(\overline{x}, \overline{y})\}\rangle \colon \top\}$ represents its effective quotient
- ($\Leftarrow$) If $\phi(\overline{x}, \overline{y})$ is *any* formula, then one may define an equivalence formula $\widehat{\phi}(\overline{x}, \overline{x}') = \forall_{\overline{y}} \phi(\overline{x}, \overline{y}) \leftrightarrow \phi(\overline{x}', \overline{y})$ and represent $\{\overline{y} \colon \phi(\overline{x}, \overline{y})\}$ by an imaginary element of $\widehat{\phi}(\overline{x}, \overline{x}')$

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Ultimate theory

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comment: $I' \subseteq I$ store consecutive approximations to t.c. of ϕ
$I' \leftarrow \emptyset$
$I \leftarrow \{\langle \overline{x}, \overline{x} \rangle : \psi(\overline{x})\}$
while $I' \neq I$ **do**
  $I' \leftarrow I$
  $I \leftarrow I \cup \{\langle \overline{x}, \overline{y} \rangle : \exists_{\overline{z}} \langle \overline{x}, \overline{z} \rangle \in I \wedge \phi(\overline{z}, \overline{y})\}$
end while
return I

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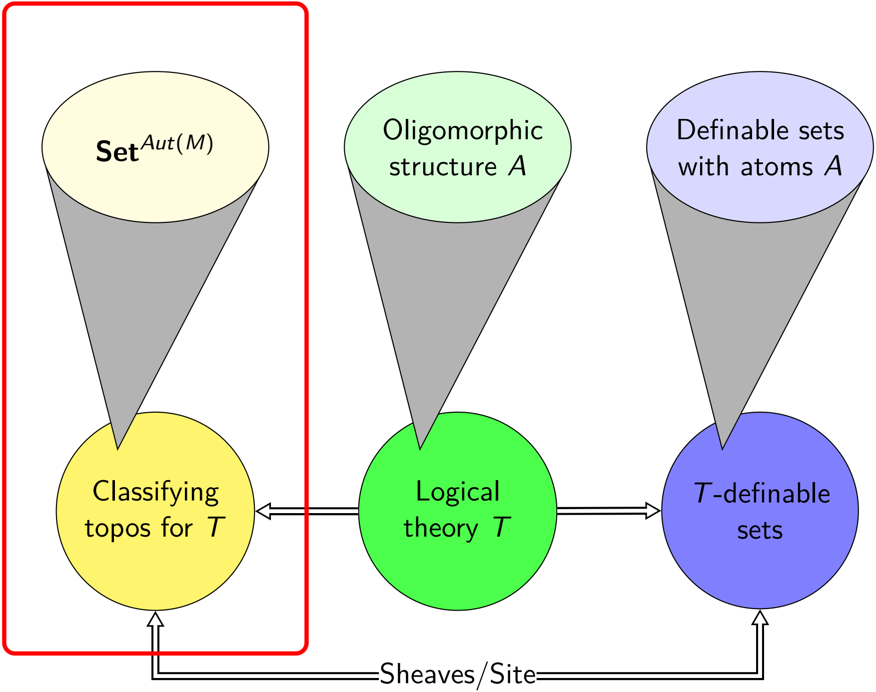
Pretopos

Every category with finite limits, existential quantifiers and well-behaved unions is equivalent to the category of T-definable sets for some theory T. Moreover, we can inject finite disjoint coproducts and effective quotients into such category making it equivalent to the category of T^+-definable sets.



$\mathbf{Set}^{Aut(M)}$
Oligomorphic structure A
Definable sets with atoms A
Classifying topos for T
Logical theory T
T-definable sets
Sheaves/Site

Bojanczyk et al. (2011)

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- For every geometric theory T there is a Grothendieck topos $C[T]$ with a generic model $\mathcal{G}_T$ of T.
- Every Grothendieck topos arises in this way

- What is a Grothendieck topos?
  - A category that behaves like the category of sets and functions in intuitionistic logic
  - A topos with small coproducts and small generating family

Classifying topos
Generic model

- What is a Grothendieck topos?
  - A category that behaves like the category of sets and functions in intuitionistic logic
  - A topos with small coproducts and small generating family
- What is a generic model of T?
  - It is a model $\mathcal{G}_T$ of T in the classifying topos $C[T]$, such that *every* model M_T of T in *any* Grothendieck topos $\mathbb{S}$ can be obtained from $\mathcal{G}_T$ in a canonical way
  - In particular, $M_T \approx F^*(\mathcal{G}_T)$, where F^* is the inverse image part of some geometric morphism $F\colon C[T] \to \mathbb{S}$

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- Examples: finite sets in **Set**, sets with finitely many orbits in $\mathbf{Set}^G$ for a coherent topological group G

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Fact:

For every first-order theory T, one may construct a Grothendieck topos $C[T]$, such that the full subcategory of $C[T]$ consisting of coherent objects is equivalent to the category of T^+-definable sets.

Beyond classifying toposes

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  - (finite) products of coherent toposes are coherent toposes
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Thank you!

Additional materials: `www.mimuw.edu.pl/~mrp/beyond.pdf`