# No more basis

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10.03.2019

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**Abstract.** The aim of this note is to show that a very weak form of Axiom of Determinacy (i.e. every word game is determined) implies that vector space $2^{\mathcal{N}}$ over the field of characteristic 2 has no basis.

## 1 The first attempt

We prove the claim by contradiction. Let us assume that $(\alpha_i)_{i \in I}$ is a basis for $2^{\mathcal{N}}$ indexed by some set $I$. Define a constant-on-basis function:

$$f(\alpha_i) = 1$$

and extend it by linearity to $2^{\mathcal{N}} \to 2$. Observe that since any $v \in 2^{\mathcal{N}}$ is a finite linear combination of some vectors from basis $(\alpha_i)_{i \in I}$ (by the definition of the basis) and the only non-zero scalar in the field of characteristic two is 1, vector $v$ may be uniquely expressed as:

$$v = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k}$$

and, thus:

$$f(v) = f(\alpha_{i_1}) + f(\alpha_{i_2}) + \cdots f(\alpha_{i_k}) = \begin{cases} 0 & \textit{if } k \textit{ is even} \\ 1 & \textit{if } k \textit{ is odd} \end{cases}$$

Consider the following game. There are two players, who take their turns in alternation. A player on its turn $i$ chooses a finite binary sequence $w_i \in 2^*$. After infinitely many turns sequences $w_i$ compose into an infinite sequence $w^\infty = w_0 w_1 w_2 \cdots \in 2^{\mathcal{N}}$. The first player wins iff $f(w^\infty) = 1$. Let us try to imitate the proof from the lecture to show that the second player does not have a winning strategy. For contrary, assume that the second player has a winning strategy. Therefore if we play $0 \in 2^*$ in the first move, the second player will win by responding $w_1 \in 2^*$. Now, imagine a second play, where we play $1w_1 \in 2^*$ as our first move. Then, because the second player has a winning strategy, he will respond by a winning move $w_2 \in 2^*$, which we may copy as our second move in the first play. And so on. In the first case, we obtain a sequence $w^\infty = 0w_0w_1w_2\cdots$, whereas in the second play we obtain the sequence $e_1 + w^\infty = 1w_0w_1w_2\cdots$, where $e_1 = 1000\cdots$. But $f(e_1 + w^\infty) = f(e_1) + f(w^\infty)$ and the second player wins in both plays iff both $f(w^\infty) = 0$ and $f(e_1) = 0$. Unfortunately, we cannot finish our argument here, because in the way we defined $f$ on arbitrary basis, $f(e_1)$ may be equal to zero.

## 2 The second attempt

Fortunately, it is easy to fix the argument. Given any basis $(\alpha_i)_{i \in I}$ for $2^{\mathcal{N}}$, decompose $e_1$ in the basis as:

$$e_1 = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k}$$

We have that:

$$\beta_l = \begin{cases} \alpha_l \textit{ if } l \neq i_1 \\ e_1 \textit{ if } l = i_1 \end{cases}$$

is another basis for $2^{\mathcal{N}}$. Define linear functional $f \colon 2^{\mathcal{N}} \to 2$ on basis $(\beta_l)_{l \in I}$:

$$f(\beta_l) = 1$$

Now, by definition $f(e_1) = 1$, therefore by the argument as in our first attempt, the second player cannot win both of the plays, therefore cannot have a winning strategy. Let us try to show, that neither the first of the players can have a winning strategy. For contrary, assume that the first player has a winning strategy and plays $w_0 \in 2^*$ in the first move. The second player plays $0 \in 2^*$, which is responded by $w_1 \in 2^*$. Now, imagine a second play, where to the first move $w_0 \in 2^*$ we play $1w_1 \in 2^*$. Then, because the first player has a winning strategy, he will respond by a winning move $w_2 \in 2^*$, which we may copy as our third move in the first play. And so on. In the first case, we obtain a sequence $w^\infty = w_0 0 w_1 w_2 \cdots$, whereas in the second play we obtain the sequence $e_k + w^\infty = w_0 1 w_1 w_2 \cdots$, where $e_k$ is the sequence consisting of all zeros but the $k$-th position, which is equal to one. But $f(e_k + w^\infty) = f(e_k) + f(w^\infty)$ and the first player wins in both plays iff both $f(w^\infty) = 1$ and $f(e_k) = 0$. Unfortunately, we cannot finish our argument here :-(, because in the way we defined $f$ on basis $(\beta_l)_{l \in I}$, $f(e_k)$ may be equal to zero. However, it should be clear, what we have to do to fix the gap in the reasoning. Given any basis $(\alpha_i)_{i \in I}$ for $2^{\mathcal{N}}$, by induction, transform it to a new basis $(\gamma_i)_{i \in I}$ that contains all standard vectors $(e_k)_{k \in \mathcal{N}}$ and define $f \colon 2^{\mathcal{N}} \to 2$ as the unique linear functional such that $f(\gamma_i) = 1$.

## 3 Yes, we have finished, but...

From any basis $(\alpha_i)_{i \in I}$ for $2^{\mathcal{N}}$ we can construct an indetermined word game, whose outcome does not depend on any *finite* number of moves. Here is an explicit construction. By induction let us transform our basis to a new basis $(\lambda_i)_{i \in I}$ that contains all standard vectors $(e_k)_{k \in \mathcal{N}}$ together with "all-ones" vector $1^\infty$. Define linear functional $f \colon 2^{\mathcal{N}} \to 2$ on basis as follows:

$$f(\lambda_i) = \begin{cases} 0 \textit{ if } \lambda_i = e_k \textit{ for some } k \in \mathcal{N} \\ 1 \textit{ otherwise} \end{cases}$$

By definition, the value of $f$ on $v$ does not depend on any finite prefix of $v$. Let us show that the word game with winning condition $f$ is indetermined. For contrary, let us assume that the first of the players has a winning strategy and starts with move $w_0 \in 2^*$. The second player responds with $0 \in 2^*$, and the first player continues with $w_1 \in 2^*$. Now, imagine a second play, where to the first move $w_0$ the second player responds with $1\overline{w_1} \in 2^*$, where $\overline{w_1}$ is the pointwise negation of $w_1$. The first player continues with $w_2 \in 2^*$, which, after negation, we copy to the first play. And so on. In the first play, we obtain a sequence $w^\infty = w_0 0 w_1 \overline{w_2} w_3 \cdots$, whereas in the second play, we obtain a sequence $w'^\infty = w_0 1 \overline{w_1} w_2 \overline{w_3} \cdots$. Observe that:

$$w^\infty + 1^\infty = \overline{w_0} 1 \overline{w_1} w_2 \overline{w_3} \cdots$$

thus:

$$w^\infty + 1^\infty + \sum_{k=1}^{|w_0|} e_k = w'^\infty$$

Therefore:

$$f(w'^\infty) = f(w^\infty) + f(1^\infty) + \sum_{k=1}^{|w_0|} f(e_k) = f(w^\infty) + 1$$

and the first player must lose in exactly one of the plays. A symmetric argument shows that the second player cannot have a winning strategy either.