# On amenability of constraint satisfaction problems

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## Abstract

The authors of [22] showed that a constraint satisfaction problem (CSP) defined over rational numbers with their natural ordering has a solution if and only if it has a definable solution. Their proof uses advanced results from topology and modern model theory. The aim of this paper is threefold. (1) We give a simple purely-logical proof of their theorem and show that the advanced results from topology and model theory are not needed; (2) we introduce an intrinsic characterisation of the statement "definable CSP has a solution iff it has a definable solution" and investigate it in general intuitionistic set theories (3) we show that the results from modern model theory are indeed needed, but for the implication reversed: we prove that "definable CSP has a solution iff it has a definable solution" holds over a countable structure if and only if the automorphism group of the structure is extremely amenable.

*CCS Concepts* • **Theory of computation → Constraint and logic programming**;

***Keywords*** set theory with atoms, intuitionistic set theory, constraint satisfaction problem, Ramsey property, extremely amenable group, Boolean prime ideal theorem

## 1 Introduction

In 1964 James D. Halpern [14] by using some combinatorial properties of the ordered Fraenkel-Mostowski model of set theory with atoms solved a long-standing open problem about independence of the Axiom of Choice from the Boolean Prime Ideal Theorem. In 2015 Bartek Klin, Eryk Kopczynski, Joanna Ochremiak, and Szymon Torunczyk [22] by using advanced results from topology and modern model theory, proved that in the ordered Fraenkel-Mostowski model of set theory with atoms an equivariant (constrained[1]) locally finite constraint satisfaction problem has a solution if and only if it has an equivariant solution. In this paper we prove that these two results are essentially the same, and, in fact, equivalent to many other well-known axioms/theorems of Boolean Set Theories. The assertion of Booleaness of the Set Theory (i.e. that the law of excluded middle holds inside the theory) is crucial for our proof, as one can find counterexamples to the claim in Intuitionistic Set Theories.

These highly-theoretical results are of a great practical interest. Very many real-world decision problems of high computational complexity can be abstractly specified as the classical constraint satisfaction problems (CSP): hardware verification and diagnosis: [9], [12], automated planning and scheduling [10], [11], temporal and spatial reasoning [31], [7], air traffic managment [1], to name a few. Such problems are inherently finite. Although their computational cost is high (i.e. the problems are usualy NP-hard), they can be solved in a finite time by a machine[2]. This is in contrast with problems concerning behaviours of autonomous systems, where the classical variant of CSP is too restrictive. Such problems can be naturally *specified* as CSP with infinite sets of variables (corresponding to the states of a system) and infinite sets of constraints (corresponding to the transitions between the states of a system). These problems are, in general, undecidable — no machine can solve them in a finite time. In fact, depending on the choice of the specification language, such problems may be very high in the undecidability hierarchy. For example, if we consider problems definable in the First-Order theory of natural numbers, then every problem from the arithmetical hierarchy can be expressed as a definable CSP. Up until recently, we had known very little about methods that can be used to solve infinite CSP. The first breakthought was at the begining of the century (see [8] and also a survey article [6]), where researches applied algebraic and model-theoretic tools to analyze CSP over, so called, infinite templates. This research inspired the Warsaw Logical Group to investigate *locally finite CSP* — i.e. infinite CSP whose constraints are finite relations (see [22] and [27]). They found that a locally finite CSP defined in the theory of rational numbers with their natural ordering can be solved effectively.

**Example 1.1** (Finite memory machine)**.** *An important type of autonomyous systems has been defined by Kaminski, Michael and Francez [20]. The authors called these type of systems "finite memory machines", or "register machines". A finite memory machine is a finite automaton augmented with a finite number of registers $R_i$ that can store natural numbers. The movement of the machine can depend on the control state, on the letter and on the content of the registers. The dependency on*

[1]Every variable appears in at least one constraint.

[2]For a general reference on solving classical CSP see: [32]

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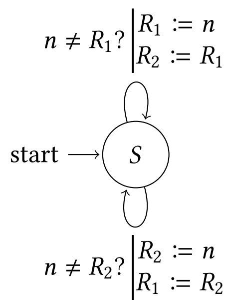


**Figure 1.** A non-deterministic machine with two registers and a single control state $S$.

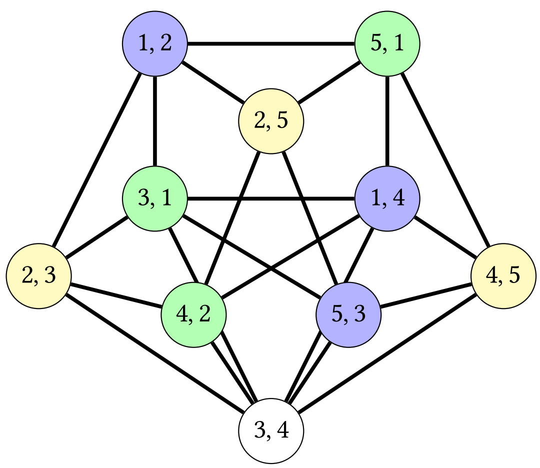


**Figure 2.** A finite counterexample to 3-colorability of infinite Kneser graph.

*the content of the registers is, however, limited — the machine can only test for equality (no formulas involving successor, addition, multiplication, etc. are allowed).*

*Figure 1 shows a 2-register machine with a single control state $S$. This machine starts with a given content of the registers $R_1$ and $R_2$ and then at every step non-deterministically chooses a register $R_i$ and a natural number $n$ such that $n \neq R_i$. Value $n$ is then stored in register $R_i$, whilst register $R_{1-i}$ gets the value previously stored in $R_i$.*

*Observe that in contrast to finite automata, the graph of possible configurations in the machine from Figure 1 is infinite. If we run the machine with $R_i := i$, then its states $V$ (i.e. single control state $S$ together with the content of the registers) will span an infinite graph $\mathcal{G} = \langle V, E \rangle$ with edges $E$ induced by the movements of the machine:*

$$\begin{aligned} V &= \{(S, n, m) \colon n \in R_1, m \in R_2\} \\ E &= \{((S, n, m), (S, n', m')) \in V \times V \colon (n = m' \wedge m \neq n') \\ &\qquad\qquad \vee (n \neq m' \wedge m = n')\} \end{aligned}$$

*This graph is known as infinite Kneser graph [22]. An example of a (constrained) locally finite CSP problem is the question whether a graph like $\mathcal{G}$ is 3-colorable. By the compactness of the first-order logic $\mathcal{G}$ is 3-colorable if and only if every finite subgraph of $\mathcal{G}$ is 3-colorable. Figure 2 exhibits an example of a finite subgraph of $\mathcal{G}$ which cannot be colored by three colors $\{red, green, blue\}$. Theorem 19 in [22] implies that 3-colourability of any graph generated by finite memory machines can be solved effectively.*

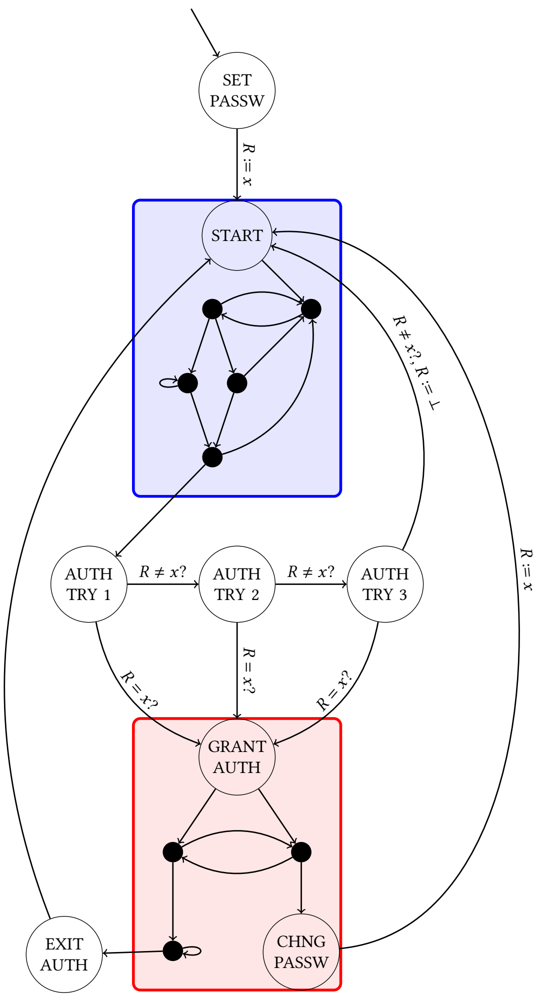


**Figure 3.** A register machine that models access control to some parts of the system.

**Example 1.2** (Access-control register machine)**.** *Continuing Example 1.1, Figure 3 presents an example of a finite memory machine with one register $R$, whose task is to model access control to the red part of the system. The machine starts in control state "SET PASW", where it awaits for the user to provide a password $x$. This password is then stored in register $R$, and the machine enters control state "START". Inside the blue rectangle the machine can perform actions that do not require* *authentication, whereas the actions that require authentication are presented inside the red rectangle. The red rectangle can be entered by the control state "GRANT AUTH", which can be accessed from one of three authentication states. In order to authorise, the machine moves to control state "AUTH TRY 1", where it gets input x from the user. If the input is the same as the value previously stored in register R, then the machine enters control state "GRANT AUTH". Otherwise, it moves to control state "AUTH TRY 2" and repeats the procedure. Upon second unsuccessful authorisation, the machine moves to control state "AUTH TRY 3". But if the user provides a wrong password when the machine is in control state "AUTH TRY 3", the register R is erased (replaced with a value that is outside of the user's alphabet) — preventing the machine to reach any of the control states from the red rectangle. Inside the red rectangle any action that requires authentication can be performed. For example, the user may request the change of the password. Example of problems that we may like to ask, which can be solved effectively:*

- *is possible to change the password in the system without exiting the blue zone?*
- *is it true that from every state from a blue zone we can enter a state in a red zone and move back (without changing the password)?*
- *is every control state in the system reachable with any content of the register?*

The main theoretical tool of the work on locally finite CSP is Theorem 17 in [22], which says that a (constrained) locally finite CSP defined in the theory of rational numbers with their natural ordering has a solution if and only if it has a definable solution. Because definable solutions over rational numbers admit exhaustive search, these CSP problems can be solved effectively. As the authors remarks (Remark 20 in [22]) their Theorem 17 can be generalised to any (decidable) relational structure with the following two properties (see Section 2 for the explanation): (a) the structure is $\omega$-categorical and (b) the automorphism group of the structure is extremely amenable. It has been further observed in [23] (an explicit reduction is given in Section 4) that definable CSP (in a finite signature) can be effectively reduced to definable CSP over finite domain. For this reason, without loss of generality, we shall focus on CSP over finite domains. Note, however, that the proofs of our Lemma 3.3 also works for a slightly more general setting of locally finite CSP.

**Example 1.3** (Rational numbers with ordering). *Let $\mathcal{Q} = \langle Q, \leq \rangle$ be the structure whose universe is interpreted as the set of rational numbers $Q$ with a single binary relation $\leq\ \subseteq Q \times Q$ interpreted as the natural ordering of rational numbers. Then the first order theory of $\mathcal{Q}$ is $\omega$-categorical, i.e. there is exactly one countable model of the theory up to an isomorphism. Moreover, the topological group of automorphism $Aut(\mathcal{Q})$ (with Tychonoff topology) is extremely amenable[3] by the main theorem in [28].*

**Example 1.4** (Rational numbers with finitely many constants). *Let $\mathcal{Q} \sqcup Q_0$ be the structure from Example 1.3 over an extended signature consisting of all constants $q \in Q_0$ for some* finite *$Q_0 \subseteq Q$ . Like in the previous example, the first order theory of $\mathcal{Q} \sqcup Q_0$ is $\omega$-categorical and the topological group of automorphisms $Aut(\mathcal{Q} \sqcup Q_0)$ is extremely amenable.*

**Example 1.5** (Ordered vector space). *Let $H_F$ be the free $\aleph_0$-dimensional vector space over a finite field $F$. By definition $H$ has a base that can be enumerated by any countable set. Therefore, we can assume that there is a base $\langle \alpha_q \rangle_{q \in \mathcal{Q}}$ enumerated by rational numbers[4] $q \in \mathcal{Q}$. In fact we can give an explicit description of space $H_F$ with its standard base as follows. Let us identify $H_F$ with a subspace of $F^{\mathcal{Q}}$ consisting of functions that have finite support — i.e. functions $v\colon Q \to F$ with the property that the set $\{q \in Q\colon v(q) \neq 0\}$ is finite. The standard base $\langle \alpha_q \rangle_{q \in Q}$ for $H_F$ consists of unit-mass functions:*

$$\alpha_q(r) = \begin{cases} 1 & \text{if } q = r \\ 0 & \text{otherwise} \end{cases}$$

*Observe, that for any vector $v \in H_F$ the evaluation $v(q) \in F$ is the $q$-th coordinate of $v$ according to the standard base. Let us choose any linear ordering $\leq_F\ \subseteq F \times F$ of the field $F$ such that $0 \in F$ is the least element in this ordering – i.e. $\forall_{r \in F}\ 0 \leq_F r$. The ordering of $F$ can be extended along the standard base to a linear ordering of $H_F$ in the following way. Consider a pair of distinct vectors $v \neq w \in H_F$ define the set $D(v, w) = \{q \in \mathcal{Q}\colon w(q) \neq v(q)\}$ of distinct coordinates between $v$ and $w$. By the definition of the base, $D(v, w)$ is finite (because every vector has only finitely many non-zero coordinates in any base) and non-empty (because $v$ and $w$ are distinct), thus it contains the largest element $m \in D(v, w)$. We shall set $v < w$ if $v(m) < w(m)$ and for general $v, w \in H_F$ define $v \leq w$ on $H_F$ as $v = w \vee v < w$.*

*Consider the structure $\mathcal{H}_F = \langle H_F, +, (-)r, \leq \rangle$, with universe $H_F$, relation $\leq\ \subseteq H_F \times H_F$ defined above, binary operation $+\colon H_F \times H_F \to H_F$ interpreted as the addition of vectors, and for each scalar $r \in F$ an unary operation $(-)r\colon H_F \to H_F$ interpreted as the multiplication $vr$ of vectors $v \in H_F$ with scalar $r$. We shall call this structure the ordered vector space over field $F$.*

*The first order theory $Th(\mathcal{H}_F)$ of $\mathcal{H}_F$ is $\omega$-categorical and the topological group of automorphisms $Aut(\mathcal{H}_F)$ is extremely amenable (see [21]).*

[3] See Section 2 Definition 2.3.

[4] A careful reader may wonder why we have indexed the base with rational numbers instead of, say, natural numbers. The reason is that we need a dense ordering on base vectors, and this can be canonically induced by the ordering of the rational numbers. If we used the ordering of the natural numbers, we would get a non-dense ordering on base vectors, and the resulting structure would not have been $\omega$-categorical, nor its automorphism group would be extremely amenable.

**Algorithm 1** K-colorability

```
procedure IS-K-COLORABLE?(⟨V, E⟩, k)
    K ← {1, 2, ..., k}
    C ← V × K
    for F ⊆ C do
        if isValid(F, E) then return ⊤
    return ⊥
procedure isValid(F, E)
    for x ∈ V do
        if F[x] = ∅ then return ⊥
        for y ∈ E[x] do
            if F[x] ∩ F[y] ≠ ∅ then return ⊥
    return ⊤
```

**Example 1.6** (Pure sets)**.** *Let $\mathcal{N} = \{0, 1, 2, \dots\}$ be a countably infinite set over empty signature $\Xi$. Then the first order theory of $\mathcal{N}$ is $\omega$-categorical, i.e. there is exactly one countable model of the theory up to an isomorphism. This theory is called the theory of "pure sets". Although the automorphism group $\mathbb{S}^\infty = Aut(\mathcal{N})$ of $\mathcal{N}$ is not extremely amenable, $\mathcal{N}$ is a reduct of $\mathcal{Q} = \langle Q, \leq \rangle$ (just drop the comparison relation). Therefore every specification in the theory of pure sets can be treated as a specification in the theory of rational numbers with inequality.*

In the language of set theory with atoms, definable sets correspond to equivariant sets of finitary type (i.e. equivariant sets that are hereditarily of a finite support, a concept that will be explained in Section 2). Moreover, a careful inspection of Theorem 17 reveals that the assumption that the sets are of a finitary type is not needed for the proof (it is needed in Theorem 19). Therefore, Theorem 17 can be restated as follows: "in the ordered Fraenkel-Mostowski model of set theory with atoms a (constrained, locally finite) equivariant constraint satisfaction problem has a solution if and only if it has an equivariant solution".

Algorithm 1 gives a method to solve the 3-colouring problem over $\omega$-categorical structures whose group of automorphism is extremely amenable. The algorithm takes for its input an equivariant definable graph consisting of nodes $V$ and edges $E$, and a number of $k$ colors. Then for every equivariant definable relation $C : V \nrightarrow K$ the algorithm tests if $C$ is functional and satisfy the colouring constraints. Upon a positive test the algorithm returns $\top$. If no relation tests positively, the algorithm returns $\bot$. The assumption of $\omega$-categoricity of the structure is crucial for the effectiveness of the *for loops* — because $V$ is definable and $k$ is finite, there are only finitely many equivariant definable subsets of $V \times K$. On the other hand, the assumption that the automorphism group is extremely amenable allows us to restrict to definable colourings only.

The above considerations beget question about necessity of properties (a) and (b) for the effectiveness of the algorithms. The case of $\omega$-categoricity has been studied in the detail in [29]. Theorem 2.5 in [29] shows that we cannot drop the assumption of $\omega$-categoricity without sacrificing effectiveness of computations. Moreover, for non-$\omega$-categorical structures we lose the correspondence between definable sets in set theories with atoms and sets definable in the structures (also see [29]). The case of extremal amenability is studied in this paper. Although we will not show that effectiveness of algorithms to CSP is equivalent to extremal amenability of the group of automorphism of the structure (what is, obviously, not true), it will turn out that it is equivalent to the effectiveness of the *natural* algorithms like Algorithm 1.

One may also wonder, what would happen if we did not restrict specification languages to complete first-order theories (i.e. to theories of algebraic structures). In the context of effectiveness of the algorithms, non-complete first-order theories (and, generally, they positiv-existential fragments) are studied in [29]. Such theories either have finitely many completions, in which case working inside the theory is equivalent to working in finitely many completions of the theory, or are classified by Intuitionistic Set Theories. As we shall see later, CSP problems in Intuitionistic Set Theories are very subtle and require different tools to analyse.

The structure of the paper is as follows. In the next section we recall some notations and results from set theory and model theory, which are necessary to understand our theorems from Section 3. In Section 3 we reformulate the property "symmetric CSP over finite domain has a solution iff it has a symmetric solution" as an intrinsic property of a topos and call it Axiom CSP. Then, we show that in Boolean toposes Axiom CSP is actually equivalent to BPIT. In particular, for every set of atoms $\mathcal{A}$ we have that $\mathbf{ZFA}(\mathcal{A})$ satisfies Axiom CSP if and only if it satisfies BPIT (Theorem 3.4). We also show that "symmetric CSP over finite domain has a solution iff it has an symmetric solution" is equivalent to a weaker axiom: "definable CSP over finite domain has a solution iff it has a definable solution". There is, however, one caveat: for the effectivenes of computations we need a slightly stronger property: "$A_0$-equivariant definable CSP over finite domain has a solution iff it has an $A_0$-equivariant definable solution". Because equivariance is not an intrinsic property of $\mathbf{ZFA}(\mathcal{A})$, therefore we have to assert Axiom CSP in every $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0))$, and then by the transfer principle (see [30]) recover the desired property in $\mathbf{ZFA}(\mathcal{A})$ (Theorem 3.8). In Section 4 we investigate Axiom CSP in non-Boolean toposes pointing out many obstacles to the equivalence between Boolean prime ideal theorem and possible formulations of Axiom CSP. We conclude the paper in Section 5.

## 2 Set Theories with Atoms

Let $\mathcal{A}$ be an algebraic structure (both operations and relations are allowed) with universum $A$. We shall think of elements of $\mathcal{A}$ as "atoms". A von Neumann-like hierarchy

$V_\alpha(\mathcal{A})$ of sets with atoms $\mathcal{A}$ can be defined by transfinite induction [26], [13]:

- $V_0(\mathcal{A}) = A$
- $V_{\alpha+1}(\mathcal{A}) = \mathcal{P}(V_\alpha(\mathcal{A})) \cup V_\alpha(\mathcal{A})$
- $V_\lambda(\mathcal{A}) = \bigcup_{\alpha<\lambda} V_\alpha(\mathcal{A})$ if $\lambda$ is a limit ordinal

Then the cumulative hierarchy of sets with atoms $\mathcal{A}$ is just $V(\mathcal{A}) = \bigcup_{\alpha\colon Ord} V_\alpha(\mathcal{A})$. Observe, that the universe $V(\mathcal{A})$ carries a natural action $(\bullet)\colon Aut(\mathcal{A}) \times V(\mathcal{A}) \to V(\mathcal{A})$ of the automorphism group $Aut(\mathcal{A})$ of structure $\mathcal{A}$ — it is just applied pointwise to the atoms of a set. If $X \in V(\mathcal{A})$ is a set with atoms then by its set-wise stabiliser we shall mean the set: $Aut(\mathcal{A})_X = \{\pi \in Aut(\mathcal{A})\colon \pi \bullet X = X\}$; and by its point-wise stabiliser the set: $Aut(\mathcal{A})_{(X)} = \{\pi \in Aut(\mathcal{A})\colon \forall_{x\in X} \pi \bullet x = x\}$. Moreover, for every $X$, these sets inherit a group structure from $Aut(\mathcal{A})$.

There is an important sub-hierarchy of the cumulative hierarchy of sets with atoms $\mathcal{A}$, which consists of "symmetric sets" only. To define this hierarchy, we have to equip $Aut(\mathcal{A})$ with the structure of a topological group. A set $X \in V(\mathcal{A})$ is *symmetric* if the set-wise stabilisers of all of its descendants $Y$ is an open set (an open subgroup of $Aut(\mathcal{A})$), i.e. for every $Y \in^* X$ we have that: $Aut(\mathcal{A})_Y$ is open in $Aut(\mathcal{A})$, where $\in^*$ is the reflexive-transitive closure of the membership relation $\in$. A function between symmetric sets is called symmetric if its graph is a symmetric set. Of a special interest is the topology on $Aut(\mathcal{A})$ inherited from the product topology on $\prod_A A = A^A$ (i.e. the Tychonoff topology). We shall call this topology the canonical topology on $Aut(\mathcal{A})$. In this topology, a subgroup $\mathbb{H}$ of $Aut(\mathcal{A})$ is open if there is a finite $A_0 \subseteq A$ such that: $Aut(\mathcal{A})_{(A_0)} \subseteq \mathbb{H}$, i.e.: group $\mathbb{H}$ contains a point-wise stabiliser of some finite set of atoms. The sub-hierarchy of $V(\mathcal{A})$ that consists of symmetric sets according to the canonical topology on $Aut(\mathcal{A})$ will be denoted by $\mathbf{ZFA}(\mathcal{A})$ (it is a model of Zermelo-Fraenkel set theory with atoms).

**Remark 2.1.** *The above definition of hierarchy of symmetric sets is equivalent to another one used in model theory. By a normal filter of subgroups of a group $\mathbb{G}$ we shall understand a filter $\mathcal{F}$ on the poset of subgroups of $\mathbb{G}$ closet under conjugation, i.e. if $g \in \mathbb{G}$ and $\mathbb{H} \in \mathcal{F}$ then $g\mathbb{H}g^{-1} = \{g\bullet h\bullet g^{-1}\colon h \in \mathbb{H}\} \in \mathcal{F}$. Let $\mathcal{F}$ be a normal filter of subgroups of $Aut(\mathcal{A})$. We say that a set $X \in V(\mathcal{A})$ is $\mathcal{F}$-symmetric if the set-wise stabilisers of all of its descendants $Y$ belong to $\mathcal{F}$ — i.e. $Y \in^* \mathcal{F}$. To see that the definitions of symmetric sets and $\mathcal{F}$-symmetric sets are equivalent, observe first that if $\mathbb{G}$ is a topological group, then the set $\mathcal{F}$ of all open subgroups of $\mathbb{G}$ is a normal filter of subgroups. In the other direction, if $\mathcal{F}$ is a normal filter of subgroups of a group $\mathbb{G}$, then we may define a topology on $\mathbb{G}$ by declaring sets $U \subseteq \mathbb{G}$ to be open if they satisfy the following property: for every $g \in U$ there exists $\mathbb{H} \in \mathcal{F}$ such that $g\mathbb{H} \subseteq U$. According to this topology a group $\mathbb{U}$ is open iff $\mathbb{U} \in \mathcal{F}$ — just observe that for every group $\mathbb{U}$ and for every $g \in \mathbb{U}$ we have that $g\mathbb{U} = \mathbb{U}$; and if $\mathbb{H} \in \mathcal{F}$ such that $\mathbb{H} = 1\mathbb{H} \subseteq \mathbb{U}$ then by the property of the filter, $\mathbb{U} \in \mathcal{F}$.*

**Example 2.1** (The basic Fraenkel-Mostowski model)**.** *Let $\mathcal{N}$ be the structure from Example 1.6. We call $\mathbf{ZFA}(\mathcal{N})$ the basic Fraenkel-Mostowski model of set theory with atoms. Observe that $Aut(\mathcal{N})$ is the group of all bijections (permutations) on $N$. The following are examples of sets in $\mathbf{ZFA}(\mathcal{N})$:*

- *all sets without atoms, e.g. $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}, \dots\}, \dots$*
- *all finite subsets of $N$, e.g. $\{0\}, \{0, 1, 2, 3\}, \dots$*
- *all cofinite subsets of $N$, e.g. $\{1, 2, 3, \dots\}, \{4, 5, 6, \dots\}, \dots$*
- $N \times N$
- $\{\langle a, b\rangle \in N^2\colon a \neq b\}$
- $N^* = \bigcup_{k\in N} N^k$
- $\mathcal{K}(N) = \{N_0\colon N_0 \subseteq N, N_0 \textit{ is finite}\}$
- $\mathcal{P}_s(N) = \{N_0\colon N_0 \subseteq N, N_0 \textit{ is symmetric}\}$

*Here are examples of sets in $V(\mathcal{N})$ which are not symmetric:*

- $\{0, 2, 4, 6, \dots\}$
- $\{\langle n, m\rangle \in N^2\colon n \leq m\}$
- *the set of all functions from $N$ to $N$*
- $\mathcal{P}(N) = \{N_0\colon N_0 \subseteq N\}$

**Example 2.2** (The ordered Fraenkel-Mostowski model)**.** *Let $\mathcal{Q}$ be the structure from Example 1.3. We call $\mathbf{ZFA}(\mathcal{Q})$ the ordered Fraenkel-Mostowski model of set theory with atoms. Observe that $Aut(\mathcal{Q})$ is the group of all order-preserving bijections on $Q$. All symmetric sets from Example 2.1 are symmetric sets in $\mathbf{ZFA}(\mathcal{Q})$ when $N$ is replaced by $Q$. Here are some further symmetric sets:*

- $\{\langle p, q\rangle \in Q^2\colon p \leq q\}$
- $\{\langle p, q\rangle \in Q^2\colon 0 \leq p \leq q \leq 1\}$

**Example 2.3** (The second Fraenkel-Mostowski model)**.** *Let $\mathcal{S} = \langle Z^*, -, (|-|_n)_{n\in N}\rangle$ be the structure of non-zero integer numbers, with unary "minus" operation $(-)\colon Z^* \to Z^*$ and with unary relations $|-|_n \subseteq Z^*$ defined in the following way: $|z|_n \Leftrightarrow |z| = n$. We call $\mathbf{ZFA}(Z^*)$ the second Fraenkel-Mostowski model of set theory with atoms. Observe that $Aut(\mathcal{Z}^*) \approx \mathbb{Z}_2^N$, therefore the following sets are symmetric in $\mathbf{ZFA}(Z^*)$:*

- $\{\dots, -6, -4, -2, 2, 4, 6, \dots\}$
- $\{\langle x, y\rangle \in Z^* \times Z^*\colon x = 3y\}$

Observe that the group $Aut(\mathcal{A})_{(A_0)}$ is actually the group of automorphism of structure $\mathcal{A}$ extended with constants $A_0$, i.e.: $Aut(\mathcal{A})_{(A_0)} = Aut(\mathcal{A} \sqcup A_0)$. Then a set $X \in V(\mathcal{A})$ is symmetric if and only if there is a finite $A_0 \in A$ such that $Aut(\mathcal{A} \sqcup A_0) \subseteq Aut(\mathcal{A})_X$ and the canonical action of topological group $Aut(\mathcal{A} \sqcup A_0)$ on discrete set $X$ is continuous. A symmetric set is called $A_0$-equivariant (or equivariant in case $A_0 = \emptyset$) if $Aut(\mathcal{A} \sqcup A_0) \subseteq Aut(\mathcal{A})_X$. Therefore, the (non-full) subcategory of $\mathbf{ZFA}(\mathcal{A})$ on $A_0$-equivariant sets and $A_0$-equivariant functions (i.e. functions whose graphs are $A_0$-equivariant) is equivalent to the category $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0)) \subseteq \mathbf{Set}^{Aut(\mathcal{A}\sqcup A_0)}$ of continuous actions of the topological group $Aut(\mathcal{A} \sqcup A_0)$ on discrete sets.

**Example 2.4** (Equivariant sets)**.** *In the basic Fraenkel-Mostowski model:*

- *all sets without atoms are equivariant*
- *all finite subsets $N_0 \subseteq N$ are $N_0$-equivariant*
- *all finite subsets $N_0 \subseteq N$ are $(N \setminus N_0)$-equivariant*
- $N \times N, N^{(2)}, \mathcal{K}(N), \mathcal{P}_S(N)$ *are equivariant*

**Definition 2.1** (Definable set). *We shall say that an $A_0$-equivariant set $X \in \mathbf{ZFA}(\mathcal{A})$ is definable if its canonical action has only finitely many orbits, i.e. if the relation $x \equiv y \Leftrightarrow \exists_{\pi \in Aut(\mathcal{A} \sqcup A_0)}\ x = \pi \bullet y$ has finitely many equivalence classes.*

For an open subgroup $\mathbb{H}$ of $Aut(\mathcal{A})$ let us denote by $Aut(\mathcal{A})/\mathbb{H}$ the quotient set $\{\pi\mathbb{H} \colon \pi \in Aut(\mathcal{A})\}$. This set carries a natural continuous action of $Aut(\mathcal{A})$, i.e. for $\sigma, \pi \in Aut(\mathcal{A})$, we have $\sigma \bullet \pi\mathbb{H} = (\sigma \circ \pi)\mathbb{H}$. All transitive (i.e. single orbit) actions of $Aut(\mathcal{A})$ on discrete sets are essentialy of this form (see for example Chapter III, Section 9 of [24]). Therefore, equivariant definable sets are essentially finite unions of sets of the form $Aut(\mathcal{A})/\mathbb{H}$. Moreover, if structure $\mathcal{A}$ is $\omega$-categorical (Example 2.1, Example 2.2, but not 2.3), then equivariant definable sets are the same as sets definable in the first order theory of $\mathcal{A}$ extended with elimination of imaginaries [30].

The Boolean Prime Ideal Theorem (BPIT) states that every ideal in a Boolean algebra can be extended to a prime ideal (we shall recall the definitions in Section 3). It is a routine to check that BPIT follows from the Axiom of Choice [18], [17]. On the other hand, it was a long-standing open problem whether the reverse implication holds as well. In 1964 James D. Halpern [14] used the model of ZFA over the rational numbers with the canonical ordering (nowadays called ordered Fraenkel-Mostowski model $\mathbf{ZFA}(\mathcal{Q})$) to prove that the Axiom of Choice is not a consequence of BPIT in a set theory with atoms — i.e, he showd that in $\mathbf{ZFA}(\mathcal{Q})$ the Axiom of Choice fails badly, but BPIT holds. This result was later amplified in [15] to give the first proof that the Axiom of Choice is not a consequence of BPIT in ZF (without atoms).

A literal formulation of the abovementioned result is as follows. Let $\mathcal{B}$ be a Boolean algebra such that $B$ is symmetric and all Boolean algebra operations are symmetric in $\mathbf{ZFA}(\mathcal{Q})$, then $\mathcal{B}$ has a symmetric ideal in $\mathbf{ZFA}(\mathcal{Q})$. However, the main Theorem in Section 4 from [14] says much more.

**Theorem 2.1** (BPIT in $\mathbf{ZFA}(\mathcal{Q})$ (1964, James D. Halpern [14])). *Let $\mathcal{B}$ be a Boolean algebra such that $B$ is $Q_0$-equivariant and all Boolean algebra operations are $Q_0$-equivariant in $\mathbf{ZFA}(\mathcal{Q})$ for some finite $Q_0 \subset Q$. Then $\mathcal{B}$ has an $Q_0$-equivariant prime ideal.*

In other words, James D. Halpern showed that for every finite $Q_0 \subset Q$, BPIT holds in the Boolean topos $\mathbf{Cont}(Aut(\mathcal{Q} \sqcup Q_0))$ of continuous actions of the topological group $Aut(\mathcal{Q} \sqcup Q_0)$ on discrete sets.

The key tool used in James D. Halpern's proof is a combinatorial lemma about partitions of sets. It was later observed that this lemma can be distilled to carry over James D. Halpern's proof from $\mathbf{ZFA}(\mathcal{Q})$ to $\mathbf{ZFA}(\mathcal{A})$ for any structure $A$ satisfying so-called Ramsey property [19], [4].

**Definition 2.2** (Ramsey property). *A structure $\mathcal{A}$ has a Ramsey property if for every open subgroup $\mathbb{H}$ of $Aut(\mathcal{A})$, every function $f \colon Aut(\mathcal{A})/\mathbb{H} \to \{1, 2, \ldots, k\}$ and every finite set $C \subseteq Aut(\mathcal{A})/\mathbb{H}$ there is $\pi \in Aut(\mathcal{A})$ such that $f$ is constant on $g \bullet C$, i.e. there exists $0 \leq i \leq k$ such that for all $c \in C$ we have $f(\pi \bullet c) = i$.*

In 1984 Peter Johnstone [19] introduced a (seemingly) stronger axiom than BPIT (but strictly weaker than AC) and showed that it holds in $\mathbf{ZFA}(\mathcal{Q})$. He called the axiom Almost Maximal Ideal Theorem (AMIT) and raised the question if AMIT is strictly stronger than BPIT. This question was answered negatively by Andreas Blass in 1986 — Theorem 1 of [4] states that BPIT implies (therefore, is equivalent to) AMIT in ZF. In that paper Andreas Blass included a preliminary version of the theorem for set theories with atoms. As a part of the theorem (i.e. Theorem 2 in [4]), he obtained the following.

**Theorem 2.2** (BPIT in $\mathbf{ZFA}(\mathcal{A})$ (1986, Andreas Blass [4])). *Let $A$ be an algebraic structure. Then $A$ has Ramsey property if and only if for every finite $A_0 \subset A$ every $A_0$-equivariant Boolean algebra has an $A_0$-equivariant prime ideal in $\mathbf{ZFA}(\mathcal{A})$.*

In 1970 Theodore Mitchell defined a certain fixed-point property of a topological group (i.e. extremal amenability) and asked if there exists a non-trivial example of such a group [25].

**Definition 2.3** (Extremely amenable group). *A topological group $\mathbb{G}$ is called extremely amenable if its every action $(\bullet) \colon \mathbb{G} \times X \to X$ on a non-empty compact Hausdorff space $X$ has a fixed point.*

In 1975 Wojchiech Herer and Jens P. R. Christensen [16] showed that there exists a non-trivial extremely amenable group. Their construction was quite artificial raising a question if there is any "natural" example of an extremely amenable group. One answer to this question was provided in 1998 by Vladimir Pestov [28].

**Theorem 2.3** (Extremal amenability (1998, Vladimir G. Pestov [21])). *The topological group of automorphisms $Aut(\mathcal{Q})$ (with Tychonoff topology) of the rational numbers with their natural ordering $Q$ is extremely amenable.*

Perhaps the most celebrated result in topological dynamic of recent years, was the theorem of Alexander S. Kechris, Vladimir G. Pestov, and Stevo Todorcevic linking Ramsey property with extremal amenability of automorphism groups (Proposition 4.2 in [21]).

**Theorem 2.4** (Ramsey vs. extremal amenability (2005, Alexander S. Kechris, Vladimir G. Pestov, and Stevo Todorcevic [21])). *Let $A$ be a single-sorted countable algebraic structure.*

*Then $A$ has Ramsey property if and only if the topological group of automorphisms $Aut(\mathcal{A})$ (with Tychonoff topology) of $A$ is extremely amenable.*

The importance of this characterisation theorem relies on the fact that it is relatively easy to prove that a structure satisfies Ramsey property (and we had known many examples of structures satisfying Ramsey property), but the property of extremal amenability is usually much harder and difficult to prove (in fact, we had known very few examples of extremely amenable groups). Moreover, extremal amenability is seemingly much more powerful than the Ramsey property. For instance, in presence of Theorem 2.3 the groundbreaking result of James D. Halpern becomes trivial: just observe that the set of all homomorphism from a $Q_0$-equivariant Boolean algebra the two-valued Boolean algebra 2 carries a compact topology (the Tychonoff topology) and the natural action of $Aut(\mathbb{Q} \sqcup Q_0)$ is continuous. It was first observed by Andreas Blass in 2011 in [5] that Theorem 2.2 together with Theorem 2.4 give another characterisation of set theories with atoms that satisfy BPIT.

In 2015 Bartek Klin, Eryk Kopczynski, Joanna Ochremiak, and Szymon Torunczyk (Theorem 17 in [22]) proved that in the ordered Fraenkel-Mostowski model of set theory with atoms a (constrained) definable equivariant locally finite constraint satisfaction problem has a solution if and only if it has an equivariant solution. To understand this result we need to recall some definitions first.

**Definition 2.4** (Constraint satisfaction problem). *A constraint satisfaction problem (CSP) consists of a triple $\langle D, V, C\rangle$, where:*

- *$D$ is a set called the domain of the problem*
- *$V$ is the set of variables*
- *$C$ is a set of constraints of the form $\langle\langle x_1, x_2, \cdots, x_k\rangle, R\rangle$, where $x_i \in V$ and $R \subseteq D^k$*

*A solution to this problem is an assignment $S\colon V \to D$ that satisfies all constraints in $C$, i.e.: for every $\langle\langle x_1, x_2, \cdots, x_k\rangle, R\rangle \in C$ we have that $R(S(x_1), S(x_2), \cdots, S(x_k))$ holds.*

**Remark 2.2.** *Every constraint satisfaction problem can be presented as a pair of relational structures $\mathcal{V}$, $\mathcal{D}$ over a single relational signature $\Sigma$. This signature $\Sigma$ consists of a pair $\langle R, k\rangle$ for every relation $R \subseteq D^k$ from a constraint $\langle\langle x_1, x_2, \ldots, x_k\rangle, R\rangle \in C$. The interpretation of symbol $R/k \in \Sigma$ in $\mathcal{V}$ is:*

$$R^V(x_1, x_2, \ldots, x_k) \Leftrightarrow \langle\langle x_1, x_2, \ldots, x_k\rangle, R\rangle \in C$$

*and the interpretation in $\mathcal{D}$ is the relation $R$ itself. Moreover, a solution $S\colon V \to D$ to the CSP becomes a homomorphism from $\mathcal{V}$ to $\mathcal{D}$.*

**Definition 2.5** (Locally finite constraint satisfaction problem). *A constraint satisfaction problem $\langle D, V, C\rangle$ is locally finite in $\mathbf{ZFA}(\mathcal{A})$ if:*

- *the domain $D$ of the problem is definable*
- *every relation $R$ in any constraint of $C$ is finite*
- *the set of variables $V$ is a definable set in $\mathbf{ZFA}(\mathcal{A})$*
- *the set $C$ of constraints is a definable set in $\mathbf{ZFA}(\mathcal{A})$*

*Furthermore, if $V$, $C$ and $D$ are $A_0$-equivariant, then we say that the CSP is $A_0$-equivariant.*

For the validity of the next theorem, we need a simple normalisation condition on locally finite CSP. Let us call a CSP *constrained* if every element of $V$ appears in at least one constraint in $C$.

**Theorem 2.5** (Locally finite CSP over rational numbers (2015, Bartek Klin, Eryk Kopczynski, Joanna Ochremiak, and Szymon Torunczyk [22])). *An equivariant constrained locally finite CSP in $\mathbf{ZFA}(\mathbb{Q})$ has a solution if and only if it has an equivariant definable solution in $\mathbf{ZFA}(\mathbb{Q})$.*

Remark 20 of [22] allows us to extend this theorem in the following way.

**Theorem 2.6** (Locally finite CSP (2015, Bartek Klin, Eryk Kopczynski, Joanna Ochremiak, and Szymon Torunczyk [22])). *Let $\mathcal{A}$ be an algebraic structure whose group of automorphism is extremely amenable. Then an $A_0$-equivariant constrained locally finite CSP in $\mathbf{ZFA}(\mathcal{A})$ has a solution if and only if it has an $A_0$-equivariant definable solution in $\mathbf{ZFA}(\mathcal{A})$.*

As we mentioned in the introduction, we shall slightly restrict the setting to CSP over finite domains. This restriction is rather technical, as it simplifies some part of the reasoning. Note, however, that our Lemma 3.2 works for locally finite CSP without any change to the proof. We use this sightly restricted conclusion of Theorem 2.6 for our axiom.

**Axiom 1** (defCSP). *For every finite $X$, an $X$-equivariant definable CSP over finite domain has a solution if and only if it has an $X$-equivariant definable solution.*

We shall write $\mathbf{ZFA}(\mathcal{A}) \vDash$ *defCSP* to indicate that Axiom defCSP holds in $\mathbf{ZFA}(\mathcal{A})$. Notice, however, that we cannot compare the strength of Axiom defCSP to other axioms in general set theories, because Axiom defCSP is not described in the language of the set theory (without atoms). The main result of this paper is to find an intrinsic characterisation of this axiom suitable for the language of set theory (Axiom CSP) and prove that over Boolean set theories it is equivalent to BPIT. To do this, we will first sharpen Axiom defCSP by removing the definability requirement (Axiom zfaCSP) and prove that over set theories with atoms these two axioms are equivalent (Theorem 3.7).

**Axiom 2** (zfaCSP). *For every finite $X$, an $X$-equivariant CSP over finite domain has a solution if and only if it has an $X$-equivariant solution.*

Figure 4 summarizes all of the abovementioned results. On the bottom side of the figure we have logical equivalences proved in this paper (Theorem 3.4 and Theorem 3.7). In the center (from the left) we have Theorem 2.5 and Theorem 2.1, which are special cases of Theorem 2.6 and Theorem 2.2 respectively. In the middle of the top of the diagram we have Theorem 2.4. Finally, on the top-left part of the figure follows from the instantiation of Axiom CSP in a topos of continuous actions of a topological group (Example 3.1). Thanks to this equivalences we can close the loop in the diagram (all statements in the top part of the diagram are equivalent).

## 3 The axiom in Boolean toposes

In this section we will work in the internal language of a Boolean topos. A reader who is not familiar with the notion of the internal language of a topos may read the proofs as taking place in any reasonable set theory[5]. We shall be extra careful when defining set-theoretic concepts, such as finiteness, or a prime ideal. Although, in Boolean toposes many different definitions of such concepts coincide, this would not be the case for non-Boolean toposes studied in Section 4.

**Definition 3.1** (Kuratowski finiteness)**.** *Let $A$ be a set. By $K(A)$ we shall mean the sub-join-semilatice of the powerset $P(A)$ generated by singletons and the empty set. A set $A$ is Kuratowski-finite if it is the top element in $K(A)$.*

For the rest of this section we will just write "finite set" for "Kuratowski-finite set". The chief idea behind the above definition is that since a non-empty finite set $A$ can be constructed from singletons by taking binary unions, we have a certain induction principle. Let us assume that: (base of the induction) $\phi$ holds for singletons, and (step of the induction) whenever $\phi$ holds for $A_0 \subseteq A$ and $A_1 \subseteq A$ then $\phi$ holds for $A_0 \cup A_1$, then (conclusion) $\phi$ holds for $A$. For example, we can show that the Axiom of Choice internally holds for finite sets.

**Lemma 3.1** (Finite Axiom of Choice)**.** *In any Boolean topos the Axiom of Choice holds for finite objects, i.e.: every surjection $e\colon X \to Y$ onto a finite set $Y$ has a section $s\colon Y \to X$, i.e.: $e \circ s = id_Y$.*

*Proof.* Let us assume that $e\colon X \to Y$ is a surjection. Then for every finite $D$, the function $e^D\colon X^D \to Y^D$, where $e^D(h) = e \circ h$, is also a surjection. This can be proven by induction over $D$. If $D$ is the empty set, or a singleton, then the claim clearly holds. Therefore, let us assume the claim holds for finite $D_0$, $D_1$ and show that it also holds for $D_0 \cup D_1$. Since the topos is Boolean, without loss of generality, we may assume that $D_0$ and $D_1$ are disjoint. By definition, the function $e^{D_0\cup D_1}\colon X^{D_0\cup D_1} \to Y^{D_0\cup D_1}$ decomposes on disjoint $e^{D_0}\colon X^{D_0} \to Y^{D_0}$ and $e^{D_1}\colon X^{D_1} \to Y^{D_1}$ with $e^{D_0\cup D_1} = e^{D_0} \times e^{D_1}$. Because the Cartesian product of two surjections is a surjection, we may infer that $e^{D_0\cup D_1}$ is a surjection, what completes the step of the induction. Therefore, if $e\colon X \to Y$ is a surjection then for every finite $D$ we have that $e^D\colon X^D \to Y^D$ is a surjection. By setting $D = Y$, we obtain that $e^Y\colon X^Y \to Y^Y$ is a surjection, and so for every $i \in Y^Y$ there exists $h \in X^Y$ such that $e^Y(h) = i$. In particular, for $id_Y \in Y^Y$ there exists $s \in X^Y$ such that $e^Y(s) = id_Y$. But, $e^Y(s) = e \circ s$, what completes the proof. ◻

[5]This should be understood *covariantly* — our reasoning must be valid in *any* reasonable set theory.

**Definition 3.2** (Finitary relation)**.** *For sets $A, B$ we shall call $K(A\times B) \subseteq P(A\times B)$ the set of finitary relations from $A$ to $B$. A finitary relation $R$ is a partial function if it is single-valued, i.e. the following holds: $R(a,b) \wedge R(a,b') \vdash b = b'$. We will denote the set of finitary partial functions from $A$ to $B$ by $B^{\underline{A}}$.*

In a Boolean topos a subset of a finite set is finite, therefore if $A$ and $B$ are finite, then a finitary relation from $A$ to $B$ is just a relation from $A$ to $B$. In particular, there is a morphism $\gamma_0\colon K(A\times B) \to K(A)$ that assigns to a finitary relation $r \in K(A\times B)$ its domain $\gamma_0(r) \in K(A) \subseteq P(A)$.

**Definition 3.3** (Jointly-total relations)**.** *For sets $A, B$, we shall say that a subset $S \subseteq K(A\times B)$ of finitary relations from $A$ to $B$ is jointly total if every finite $A_0 \in K(A)$ is a subdomain of a finitary relation from $S$, i.e.: $\exists_{h\in S} A_0 \subseteq \gamma_0(h)$*

**Definition 3.4** (Jointly total family of homomorphisms)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two relational structures over a common signature $\Sigma$. A finitary relation from $\mathcal{A}$ to $\mathcal{B}$ preserves relation $R/k \in \Sigma$ if the following holds:*

$$f(a_1,b_1) \wedge f(a_2,b_2) \wedge \ldots \wedge f(a_k,b_k) \wedge R(a_1,a_2,\cdots,a_k) \vdash R(b_1,b_2,\cdots,b_k)$$

*We shall say that a set of finitary partial functions $H \subseteq B^{\underline{A}}$ is a jointly-total family of homomorphisms if for every finite set of relational symbols $\Sigma_0 \subseteq \Sigma$ every finite $A_0 \in K(A)$ is a subdomain of a finitary partial function from $H$ that preserves all relations from $\Sigma_0$.*

Now, we are ready to state Axiom CSP in Boolean toposes.

**Axiom 3** (CSP)**.** *For every relational signature $\Sigma$ and a pair of structures $\mathcal{V}$ and $\mathcal{D}$ over $\Sigma$ such that $D$ is a finite cardinal, the following are equivalent:*

- *there exists a homomorphism from $\mathcal{V}$ to $\mathcal{D}$*
- *the set of partial functions $D^{\underline{V}}$ is a jointly total family of homomorphisms*

By $\hom(\underline{\mathcal{V}}, \mathcal{D})$ we shall denote the set of all finitary partial functions from $A$ to $B$ that preserve *all* relations from $\Sigma$. Observe that if Axiom CSP holds then the set $\hom(\underline{\mathcal{V}}, \mathcal{D})$ is jointly total if and only if the set of of partial functions $D^{\underline{V}}$ is a jointly total family of homomorphisms.

**Example 3.1** (Axiom CSP in $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0))$)**.** *Let $\langle \Sigma, \mathcal{V}, \mathcal{D}\rangle$ be an $A_0$-equivariant CSP in $\mathbf{ZFA}(\mathcal{A})$. Such CSP is an internal object of $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0))$. The object $K(V\times D)$ of Kuratowski finite subobjects of $V\times D$ consists of finitely supported finite relations from $V$ to $D$. Because every finite set is finitely*

$$\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0)) \vDash CSP \iff \mathbf{ZFA}(\mathcal{A}) \vDash defCSP \Leftarrow Aut(\mathcal{A}) \text{ is extr. ame.} \Leftrightarrow \mathcal{A} \text{ is Ramsey} \Leftrightarrow \mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0)) \vDash BPIT$$

$$\mathbf{ZFA}(\mathcal{Q}) \vDash defCSP \Leftarrow Aut(\mathcal{Q}) \text{ is extr. ame.} \qquad \mathcal{Q} \text{ is Ramsey} \Rightarrow \mathbf{Cont}(Aut(\mathcal{Q} \sqcup Q_0)) \vDash BPIT$$

$$\mathbf{ZF} \vDash CSP \leftrightarrow BPIT \qquad \mathbf{ZFA} \vDash defCSP \leftrightarrow zfaCSP$$

**Figure 4.** Equivalences between Ramsey property, extremal amenability, BPIT and Axiom CSP.

*supported (by the union of supports of its elements), $K(V \times D)$ consists of all finite relations from $A$ to $B$. This means that the internal object of finitary homomorphisms* $\hom(\underline{\mathcal{V}}, \mathcal{D})$ *consists of the set of all finitary homomorphisms from $\mathcal{V}$ to $\mathcal{D}$. By Axiom CSP in the real world*[6] *this set is jointly total if and only if $\langle \Sigma, \mathcal{V}, \mathcal{D} \rangle$ has a solution (in* **Set***). Therefore, Axiom CSP in* $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0))$ *states that an $A_0$-equivariant CSP over a finite domain has an $A_0$-equivariant solution if and only if it has a solution (in* **Set***).*

To state our main results, we have to recall more set-theoretic terminology.

**Definition 3.5** (Boolean algebra)**.** *An algebra $\mathcal{B}$ is a structure $\langle B, 1, \wedge, \neg \rangle$, where $1$ is a constant, $\wedge\colon B \times B \to B$ is a binary operation, and $\neg\colon B \to B$ is an unary operation. Consider relation $\leq \subseteq B \times B$ defined as: $a \leq b \Leftrightarrow a = a \wedge b$. We say that $\mathcal{B}$ is a Boolean algebra if the following holds:*

- *$\leq$ is a partial order on $B$ with finite joins given by $\wedge$ and the greatest element $1$*
- *for every $b \in B$ we have that: $\neg\neg b = b$*

If $\mathcal{B}$ is a Boolean algebra, then $1$ is its internal true value, and operation $\wedge$ is the internal conjunction. Other operations in a Boolean algebra can be defined in the usual way:

- $0 = \neg 1$ for the false value
- $a \vee b = \neg(\neg a \wedge \neg b)$ for the internal disjunction
- $a \oplus b = (a \wedge b) \vee (\neg a \wedge \neg b)$

**Definition 3.6** (Ideal)**.** *Let $\mathcal{B}$ be a Boolean algebra. An ideal in $\mathcal{B}$ is a proper subset $I \subset B$ satisfying the following conditions:*

- *if $a, b \in I$ then $a \vee b \in I$*
- *if $a \in I$ then for every $b \in B$ such that $b \leq a$ we have that $b \in I$*

**Definition 3.7** (Prime ideal)**.** *Let $I$ be an ideal in $\mathcal{B}$. We say that $I$ is prime if for every $b \in B$ either $b \in I$ or $\neg b \in I$.*

**Axiom 4** (BPIT)**.** *Every ideal in a Boolean algebra can be extended to a prime ideal*

**Remark 3.1.** *An ideal $I$ in $\mathcal{B}$ can be represented by a homomorphism $h_I\colon \mathcal{B} \to \mathcal{B}'$ to a Boolean algebra $\mathcal{B}'$, i.e. $I = h_I^{-1}(0)$. A prime ideal is an ideal that can be represented by a homomorphism to $2$ equipped with the usual Boolean algebra structure. Therefore, an ideal $I$ in $\mathcal{B}$ can be extended to a prime ideal $P$ iff $\mathcal{B}'$ has a prime ideal $J$. In this case, $h_P = h_J \circ h_I$. This means, that BPIT is equivalent to the statement that every non-trivial Boolean algebra has a prime ideal. We shall use this characterisation for the reminder.*

The constraint satisfaction problem is defined over *relational* structures. Therefore, to fit into the framework of CSP we should treat a Boolean algebra $\mathcal{B}$ as if it was defined over a relational signature, with an unary predicate $top(x) \Leftrightarrow x = 1$, ternary predicate $and(x, y, z) \Leftrightarrow x \wedge y = z$ and binary predicate $not(x, y) \Leftrightarrow \neg x = y$. The axioms must express that there exists unique $x$ that satisfy *top* and that *and* and *not* are functional relations.

**Lemma 3.2** (Axiom CSP implies BPIT)**.** *Axiom CSP implies BPIT in Boolean toposes.*

*Proof.* Let $\mathcal{B}$ be a non-trivial Boolean algebra. By Axiom CSP, it suffices to show that the set of finitary homomorphisms $\hom(\underline{\mathcal{B}}, 2)$ is jointly total, i.e. for every finite $B_0$ in $K(B)$ there exists a partial homomorphism $B_0 \subseteq B_1 \to 2$. We can assume that $B_1$ is closed under Boolean-algebra operations and still finite. The reason for that is that if $B_0$ is finite then in a Boolean topos $P(B_0)$ is finite as well (it coincides with $K(B_0)$). Because AC holds for finite sets (Lemma 3.1), $B_1$ has a prime ideal, what completes the proof. □

Let us work out the proof of Theorem 3.2 in $\mathbf{Cont}(Aut(\mathcal{A}))$. For a Boolean algebra $\mathcal{B}$ treated as an object in $\mathbf{Cont}(Aut(\mathcal{A}))$ the object of finitary homomorphisms $\hom(\underline{\mathcal{B}}, 2)$ consists of finitely supported homomorphisms from a finitely supported finite subsets of $\mathcal{B}$ to $2$, i.e.:

$$\hom(\underline{\mathcal{B}}, 2) = \{h\colon B_0 \to 2 \colon B_0 \subseteq B, B_0, h \text{ are finitely supported}\}$$

Because every finite set is finitely supported, and every function between finite sets is finitely supported the above is just the set of all finitary homomorphisms from $\mathcal{B}$ to $2$, i.e.:

$$\hom(\underline{\mathcal{B}}, 2) = \{h\colon B_0 \to 2 \colon B_0 \subseteq B\}$$

For any finite $B_0 \subseteq B$ consider the Boolean algebra $\overline{B}_1$ generated by $B_0$. Let us assume that $\mathcal{B}$ is non-trivial. Because $\overline{B}_1$ is easily seen to be finite, by BPIT for finite Boolean algebras in the real world, one can find a homomorphism $\overline{B}_1 \to 2$. Therefore, $\hom(\underline{\mathcal{B}}, 2)$ is jointly total, and by Axiom CSP, there exists a homomorphism $\mathcal{B} \to 2$.

[6] We may assume Axiom CSP holds in **Set** by Theorem 3.4.

**Lemma 3.3** (BPIT implies Axiom CSP)**.** *Axiom BPIT implies Axiom CSP in Boolean toposes.*

*Proof.* Let us assume that $\mathcal{V}$ and $\mathcal{D}$ are structures over relational signature $\Sigma$. Furthermore, assume that $\mathcal{D} = \{0, 1, \ldots, D-1\}$ is a finite cardinal and the set of partial functions $D^{\underline{V}}$ is a jointly total family of homomorphisms. We shall treat $Var = V \times D$ as a set of propositional variables. Consider the following subsets of propositions $F(Var)$, where $F(Var)$ is treated as the free Boolean algebra on $Var$:

- $T = \{\langle v, 0\rangle \vee \langle v, 1\rangle \vee \cdots \vee \langle v, D-1\rangle : v \in V\}$
- $S = \{\neg(\langle v, n\rangle \wedge \langle v, m\rangle) : v \in V, n \in D, m \in D, n \neq m\}$
- $C_R = \{\neg(\langle x_1, d_1\rangle \wedge \langle x_2, d_2\rangle \wedge \cdots \wedge \langle x_n, d_n\rangle) : R^V(x_1, x_2, \ldots, x_n) \wedge \neg R^D(d_1, d_2, \ldots, d_n)\}$ for every $R \in \Sigma$

The intuitive meaning of these formula should be obvious: formulas from $T$ say that every $v \in V$ is associated to at least one value in $D$ (i.e. the relation is total); formulas from $S$ say that every $v \in V$ is associated to at most one value in $D$ (i.e. the relation is single valued); formulas from $C_R$ say that the valuations cannot volatile constraint $R$. Consider the following set of propositions: $P = T \cup S \cup \bigcup_{R \in \Sigma} C_R$. Let us say that two propositions $\phi, \psi$ from $F(Var)$ are equivalent if there is a finite $P_0 \subset P$ such that every valuation $V_0 \times D \to 2$ satisfying $P_0$ satisfies $\phi \oplus \psi$. Then $F(Var)$ divided by this equivalence relation is again a Boolean algebra $F(Var)/{\equiv}$ with the usual operations. We want to show that $F(Var)/{\equiv}$ is non-trivial, i.e. $0 \neq 1$. It suffices to show that every finite $P_0 \subseteq P$ is satisfiable. Because every finite $P_0 \subseteq P$ involves only finitely many variables $Var_0 \subseteq Var$ and finitely many constraints, the sets $V_0 = \gamma_0(Var_0) \subseteq V$ and $\Sigma_0 \subseteq \Sigma$ of relations that appear in $P_0$ are finite. In fact, $P_0$ can be rewritten as the union of:

- $T_0 = \{\langle v, 0\rangle \vee \langle v, 1\rangle \vee \cdots \vee \langle v, D-1\rangle : v \in V_0\}$
- $S_0 = \{\neg(\langle v, n\rangle \wedge \langle v, m\rangle) : v \in V_0, n \in D, m \in D, n \neq m\}$
- $C_{R,0} = \{\neg(\langle x_1, d_1\rangle \wedge \langle x_2, d_2\rangle \wedge \cdots \wedge \langle x_n, d_n\rangle) : x_i \in V_0, R^V(x_1, x_2, \ldots, x_n) \wedge \neg R^D(d_1, d_2, \ldots, d_n)\}$ for every $R \in \Sigma_0$

Since $V_0$ and $\Sigma_0$ are finite, by the assumption of Axiom CSP, there exists a finitary partial function $h_0 \in \hom(\underline{\mathcal{V}}, \mathcal{D})$ with $V_0 \subseteq \gamma_0(h_0)$ that preserves relations from $\Sigma_0$. This partial function induces a valuation $V_0 \times D \to 2$. By the definition of the constraints, this valuation makes $P_0$ satisfiable. Therefore, every finite $P_0$ is satisfiable, and so $F(Var)/{\equiv}$ is non-trivial. By BPIT, there is a prime ideal $u : F(Var)/{\equiv} \to 2$, which composed with the canonical embedding $j : F(Var) \to F(Var)/{\equiv}$ gives a prime ideal $\overline{h} = u \circ j$ on $F(Var)$. By the definition of the quotient algebra, $\overline{h}$ maps propositions from $P$ to 1. Consider the restriction $h : V \times D \to 2$ of $\overline{h}$ to variables $Var = V \times D$. By propositions $T$ valuation $h$ is total and by propositions $S$ it is single-valued. Moreover, by propositions $C_R$ the valuation does not violate any of the constraints $R \in \Sigma$. Therefore, $h$ is a homomorphism from $\mathcal{V}$ to $\mathcal{D}$. □

**Theorem 3.4** (*CSP* $\leftrightarrow$ *BPIT*)**.** *In every Boolean topos the following are equivalent:*

- *Boolean prime ideal theorem*
- *Axiom CSP*

*Proof.* By Lemma 3.3 and Lemma 3.2. □

### 3.1 Characterisation theorems

This subsection states our main characterisation theorems. Let us begin with the observation that (a variant[7] of) Theorem 17 in [22] follows from James D. Halpern's result [14] from 1964 and our Theorem 3.4 (no advanced modern results are needed).

**Theorem 3.5** (defCSP in $\mathbf{ZFA}(\mathbb{Q})$)**.** *defCSP holds in* $\mathbf{ZFA}(\mathbb{Q})$.

*Proof.* By Theorem from Section 3 in [14] every $\mathbb{Q}_0$-equivariant Boolean algebra $\mathcal{B}$ in $\mathbf{ZFA}(\mathbb{Q})$ has an $\mathbb{Q}_0$-equivariant prime ideal. This is the same as saying that every internal Boolean algebra in the topos $\mathbf{Cont}(Aut(\mathbb{Q} \sqcup \mathbb{Q}_0))$ has an internal prime ideal. Because the topos is Boolean, by Theorem 3.4 Axiom CSP holds in $\mathbf{Cont}(Aut(\mathbb{Q} \sqcup \mathbb{Q}_0))$. Therefore, every $\mathbb{Q}_0$-equivariant CSP over a finite domain has an $\mathbb{Q}_0$-equivariant solution. In other words zfaCSP holds in $\mathbf{ZFA}(\mathbb{Q})$, and since the formulation of defCSP is weaker than zfaCSP, defCSP holds in $\mathbf{ZFA}(\mathbb{Q})$ as well. □

The proof of the next theorem is similar. We use the result of Andreas Blass from 1986 [4].

**Theorem 3.6** (zfaCSP in $\mathbf{ZFA}(\mathcal{A})$)**.** *zfaCSP holds in* $\mathbf{ZFA}(\mathcal{A})$ *if and only if* $\mathcal{A}$ *satisfies Ramsey property.*

*Proof.* Theorem 2 of [4] states that Ramsey property of $\mathcal{A}$ is equivalent to the property that every $A_0$-equivariant Boolean in $\mathbf{ZFA}(\mathcal{A})$ has an $A_0$-equivariant prime ideal. This is the same as saying that every internal Boolean algebra in the topos $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0))$ has an internal prime ideal. By Theorem 3.4 this is equivalent to Axiom CSP in $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0))$. Therefore, zfaCSP holds in $\mathbf{ZFA}(\mathcal{A})$ if and only if $\mathcal{A}$ has Ramsey property. □

We will show now that seemingly weaker Axiom defCSP is actually equivalent to Axiom zfaCSP over ZFA. We need one more definition.

**Definition 3.8** (Compact object)**.** *An object $X$ of a cocomplete category $\mathbb{C}$ is called compact if its co-representation:*

$$\hom_{\mathbb{C}}(X, -) : \mathbb{C} \to \mathbf{Set}$$

*preserves filtered colimits of monomorphisms.*

[7] As mentioned in Section 2 our Theorem 3.4 can be modified to locally finite CSP.

For any topological group $\mathcal{G}$ compact objects in $\mathbf{Cont}(\mathbb{G})$ are precisely the actions that have finitely many orbits. Therefore, compact objects in $\mathbf{Cont}(Aut(\mathcal{A} \sqcup A_0))$ are just $A_0$-equivariant definable objects in $\mathbf{ZFA}(\mathcal{A})$. Consider the following axiom that can be interpreted in any topos with filtered colimits.

**Axiom 5** (ctCSP)**.** *For every compact relational signature $\Sigma$ and a pair of structures $\mathcal{V}$ and $\mathcal{D}$ over $\Sigma$ such that $D$ is a finite cardinal and $\mathcal{V}$ is compact, the following are equivalent:*

- *there exists a homomorphism from $\mathcal{V}$ to $\mathcal{D}$*
- *the set of finitary functions* $\hom(\underline{\mathcal{V}}, \mathcal{D})$ *is a jointly total family of homomorphisms*

**Theorem 3.7** (ctCSP implies Axiom CSP in continuous sets)**.** *Let $\mathbb{G}$ be a topological group and $\mathbf{Cont}(\mathbb{G})$ be the topos of its continuous actions on* **Set***. Then ctCSP holds in $\mathbf{Cont}(\mathbb{G})$ iff Axiom CSP holds in $\mathbf{Cont}(\mathbb{G})$.*

*Proof.* An object $X$ in $\mathbf{Cont}(\mathbb{G})$ can be represented as a disjoint union of its orbits $X = \bigcup X/\mathbb{G}$. By the definition of compactness for every finite set of orbits $F \subseteq X/\mathbb{G}$ the object $X_0 = \bigcup F$ is compact.

Assume that ctCSP holds in $\mathbf{Cont}(\mathbb{G})$ and consider a CSP $\langle \Sigma, \mathcal{V}, \mathcal{D} \rangle$ such that the set of partial functions $D^{\underline{V}}$ is a jointly total family of homomorphisms. Then for any restriction of $\Sigma$ to $\Sigma_0$ and any substructure $\mathcal{V}_0 \subseteq \mathcal{V}$ interpreted over $\Sigma_0$ the set of partial functions $D^{\underline{V_0}}$ is a jointly total family of homomorphisms. Thus, if we assume that $\Sigma_0$ and $\mathcal{V}_0$ have finitely many orbits (i.e. they are compact), by ctCSP there exists an equivariant homomorphism $\mathcal{V}_0 \to \mathcal{D}$ over $\Sigma_0$.

Now, observe that every CSP $\langle \Sigma, \mathcal{V}, \mathcal{D} \rangle$ in $\mathbf{Cont}(\mathbb{G})$ can be regarded as a classical structure $\langle \widetilde{\Sigma}, \widetilde{\mathcal{V}}, \widetilde{\mathcal{D}} \rangle$ over an extended signature $\widetilde{\Sigma}$ that forces homomorphisms to be equivariant. This extended signature consists of a new ternary relation $T$ and two new unary relations $U_G, U_V$. To turn $\mathcal{V}$ into a classical structure $\widetilde{\mathcal{V}}$ we extend the sort $V$ by elements of $\mathbb{G}$, i.e. $\widetilde{V} = V \sqcup G$ and define new relations as follows:

- $U_G(x) \leftrightarrow x \in G$
- $U_V(x) \leftrightarrow x \in V$
- $T(g, a, b) \leftrightarrow U_G(g) \wedge (g \bullet a = b)$

To turn $\mathcal{D}$ into a classical structure $\tilde{\mathcal{V}}$ we set $\widetilde{D} = D \sqcup \{1\}$ and define new relations as:

- $U_G(x) \leftrightarrow x = 1$
- $U_V(x) \leftrightarrow x \in D$
- $T(g, a, b) \leftrightarrow U_G(g) \wedge (a = b)$

Then CSP $\langle \Sigma, \mathcal{V}, \mathcal{D} \rangle$ has a solution in $\mathbf{Cont}(\mathbb{G})$ if and only if $\langle \widetilde{\Sigma}, \widetilde{\mathcal{V}}, \widetilde{\mathcal{D}} \rangle$ has a solution in **Set**. By Axiom CSP in the real world (i.e. in **Set**), $\widetilde{\mathcal{V}}, \widetilde{\mathcal{D}}$ has a solution if and only if every finite substructure $\widetilde{\mathcal{V}_0}$ of $\widetilde{\mathcal{V}}$ over $\widetilde{\Sigma_0}$ has a solution. But every finite $\widetilde{\mathcal{V}_0}$ has a solution, because it is contained in some $\widetilde{\mathcal{V}_1}$ for compact $\mathcal{V}_1$. Therefore, $\widetilde{\mathcal{V}}, \widetilde{\mathcal{D}}$ has a solution, which, by definition, is an equivariant solution of $\mathcal{V}, \mathcal{D}$. ◻

We can summarize the above characterisations in the next theorem.

**Theorem 3.8** (Characterisation theorem)**.** *Let $\mathcal{A}$ be a countable structure. Then the following are equivalent:*

1. *$Aut(\mathcal{A})$ is extremely amenable*
2. *$\mathcal{A}$ has Ramsey property*
3. *zfaCSP holds in $\mathbf{ZFA}(\mathcal{A})$*
4. *defCSP holds in $\mathbf{ZFA}(\mathcal{A})$*
5. *Axiom CSP holds in $\mathbf{Cont}(Aut(\mathbb{A} \sqcup A_0))$ for every finite $A_0 \subset A$*
6. *Boolean Prime Ideal theorem holds in $\mathbf{Cont}(Aut(\mathbb{A} \sqcup A_0))$ for every finite $A_0 \subset A$*

*Proof.* (1) ⇔ (2) is Proposition 4.7 in [21]. (2) ⇔ (3) is the subject of Theorem 3.6. (3) ⇔ (4) is the consequence of Theorem 3.7. (3) ⇐ (5) is trivial, (5) ⇒ (6) is the subject of 3.4, and (1) ⇔ (6) is the subject of [5]. ◻

Here are another two important statements equivalent to BPIT over Boolean toposes, therefore after suitable augmention, equivalent to Axiom defCSP.

**Corollary 3.9.** *Let $\mathcal{A}$ be an algebraic structure. Then the following are equivalent in $\mathbf{ZFA}(\mathcal{A})$:*

- *defCSP*
- *every $A_0$-equivariant non-trivial ring with unit has an $A_0$-equivariant prime ideal [3]*
- *every $A_0$-equivariant non-trivial complete distributive lattice with compact unit has an $A_0$-equivariant prime element [2]*

# 4 The case of non-Boolean toposes

When we move to non-Boolean toposes, we have to be extra careful when stating classical definitions and axioms, because in constructive mathematics classically equivalent statements may be far different. Fortunately for us, the concept of Boolean algebra, ideal and prime ideal move smoothly to the intuitionistic setting with one caveat: not every maximal ideal in a Boolean algebra has to be prime.

On the other hand, Axiom CSP is much more difficult to handle in the intuitionistic setting. Actually, we have at least several different variants of Axiom CSP depending on our interpretation of "finiteness" and admissible relational structures. Therefore, we should not expect that Axiom CSP is equivalent to BPIT in constructive mathematics, because BPIT does not involve any notion of finiteness and there is not much concern about admissibility of Boolean algebra operations (however, we could take this into account). In general, the stronger the notion of "finiteness" and "admissibility" is, the stronger Axiom CSP we obtain.

Let us discuss some possible definitions for an admissible structure $\mathcal{A}$:

1. Only complemented relations $R$ are admissible. That is, subobjects $s\colon R \to A^k$ such that there exists a subobject $\neg s\colon R \to A^k$ with the property that $s \cup \neg s = id_{A^k}$ and $s \cup \neg s = 0$.
2. $A$ is decidable. That is, the sobobject $\Delta\colon A \to A{\times}A$ that correspond to the equality predicate is complemented. Because, we assume that equality is always presented in the signature, decidability of $A$ is subsumed by the previous point.
3. All relations are admissible.

In the next subsections we discuss Axiom CSP with respect to two internal notions of "finiteness":

- Kuratowski finiteness from Definition 3.1
- Kuratowski subfiniteness — i.e. being a subobject of a Kuratowski finite object

### 4.1 Kuratowski finiteness is too strong

Consider Sierpienski topos $\mathbf{Set}^{\bullet\to\bullet}$. It is a routine to check that BPIT holds in $\mathbf{Set}^{\bullet\to\bullet}$, but Axiom CSP does not hold even in case "finiteness" is interpreted as Kuratowski finiteness and only complemented relations $R$ are admissible. For a counterexample consider structures from Figure 5. The structure on the right side is the terminal object $1$ equipped with the empty unary relation $0 \to 1$. The structure on the left side is the only non-trivial subobject $\frac{1}{2}$ of $1$ equipped with the full unary relation $\frac{1}{2} \xrightarrow{id} \frac{1}{2}$. There is a unique morphism $!$ from $\frac{1}{2}$ to $1$, but it is not a homomorphism, since it does not preserve the unary relation, i.e. $! \circ id_{\frac{1}{2}} = \frac{1}{2} \neq 0$. On the other hand, the only Kuratowski finite subobject of $\frac{1}{2}$ is $0$ and the object of homomorphisms $\hom(0, 1)$ is isomorphic to $1$.

This example shows that Axiom CSP with Kuratowski finiteness is too strong to be provable from BPIT, and too strong in general. If we weaken Axiom CSP by weakening the notion of finiteness to Kuratowski subfiniteness then Axiom CSP will hold even if all relations are admissible. The reason is that a structure $\mathcal{X}\colon \mathcal{A} \to \mathcal{B}$ in $\mathbf{Set}^{\bullet\to\bullet}$ can be encoded as a structure $\mathcal{A} \sqcup \mathcal{B}$ in $\mathbf{Set}$ with one additional relation encoding the graph of function $X$, i.e. $R(x, y) \Leftrightarrow X(x) = y$ and one unary relation to distinguish domain from the codomain, i.e. $S(x, y) = A{\times}A {\sqcup} B{\times}B$. Then for every finite substructure $A_0 \sqcup B_0$ of $\mathcal{A} \sqcup \mathcal{B}$ there is a finite substructure $A_0 \sqcup B_0 \subset A_1 \sqcup B_1 \subseteq \mathcal{A} \sqcup \mathcal{B}$ corresponding to a Kuratowski subfinite substructure of $\mathcal{X}$. Therefore, Axiom CSP holds in $\mathbf{Set}^{\bullet\to\bullet}$ by the Axiom CSP in $\mathbf{Set}$.

### 4.2 Kuratowski subfiniteness is too weak

Consider topos $\mathbf{Set}^{\bullet\leftarrow\bullet\to\bullet}$. Figure 6 shows an example of a Boolean algebra, which does not have a prime ideal. Moreover, this example explicitly shows why we cannot carry over our proof of Theorem 3.2 to constructive mathematics — the Boolean algebra under consideration is Kuratowski finite, what means that in $\mathbf{Set}^{\bullet\leftarrow\bullet\to\bullet}$ not every finite Boolean algebra has a prime ideal. On the other hand, Axiom CSP with Kuratowski finite subobjects fails and with Kuratowski subfinite subobjects holds for the same reasons as in the Sierpienski topos $\mathbf{Set}^{\bullet\to\bullet}$. Therefore, example from Figure 6 shows that Axiom CSP with Kuratowski subfiniteness is too weak to prove BPIT.

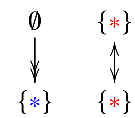

**Figure 5.** Axiom CSP fails in $\mathbf{Set}^{\bullet\to\bullet}$ for Kuratowski finiteness. The structures are equipped with a single unary relation that holds on blue elements only.

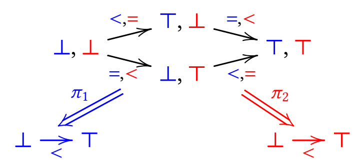

**Figure 6.** An example of a non-trivial finite Boolean algebra in $\mathbf{Set}^{\bullet\leftarrow\bullet\to\bullet}$ that has no prime ideal.

## 5 Conclusions and further work

In this paper we have given a simple purely-logical proof of "equivariant definable CSP over finite domain has a solution iff it has an equivariant definable solution" in the ordered Fraenkel-Mostowski model (Theorem 3.5) without using any advanced results from topology and model theory. Moreover, we have introduced an intrinsic characterisation of this statement and investigate it in general toposes. It turns out that in Boolean toposes this axiom is equivalent to Boolean Prime Ideal theorem, whereas in intuitionistic toposes there is no such an equivalence, nor an implication in either directions. It is an interesting question which positive-existential theories have classifying toposes validating Axiom CSP; or more generally, in which Grothendieck toposes Axiom CSP holds. Finally, we reversed the main result of [22] by showing that for a countable structure $\mathcal{A}$ Axiom defCSP holds in $\mathbf{ZFA}(\mathcal{A})$ if and only if the automorphism group $Aut(\mathcal{A})$ of $\mathcal{A}$ is extremely amenable (Theorem 3.8).

## Acknowledgments

This research was supported by the National Science Centre, Poland, under projects 2018/28/C/ST6/00417.

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