Infinite chocolate

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Abstract. The aim of this note is to describe the standard proof of the Fundamental Theorem of Chomp Game: for each number n there exists a unique ordinal number λ such that the second player has a winning strategy in Chomp game on $n \times \lambda$ chocolate.

1 Infinite chocolate

A configuration in the Chomp game on a chocolate of size $n \times \lambda$ may be represented by a tuple $\overline{a} = \langle a_1, a_2, \ldots, a_n \rangle$, where $a_i \leq \lambda$. We say that a configuration $\overline{b}$ is reachable from a configuration $\overline{a}$ if there exists a (possibly empty) sequence of chomp moves that leads to configuration $\overline{b}$ from configuration $\overline{a}$; and it is directly reachable if it can be reached in at most one move. A configuration is valid if it is reachable from the initial configuration, that is, from configuration: $\langle \lambda, \lambda, \ldots, \lambda \rangle$. It is easy to proof by induction that the valid configurations are these configurations that are weakly decreasing — i.e. $a_i \geq a_{i+1}$ for all $1 \leq i < n$. Moreover configurations that are directly reachable from the starting configuration are of the form:

$$d_k(\lambda, a) = \langle \overbrace{\lambda, \lambda, \ldots, \lambda}^{n-k}, \underbrace{a, a, \ldots, a}_{k} \rangle$$

A configuration is *winning* if the player who has to make a move from this configuration has a winning strategy. Otherwise it it a losing configuration.

2 A fool who became wise

> "The fool who persists in his folly will become wise."
> — William Blake

Very many mathematical constructions follow the "carrot and stick" pattern: a stubborn fool attracted by a carrot makes a step forward, but at the same time the carrot moves away from the fool and he gets nothing. The whole situation repeats. After any number of steps he does not have any reward for his actions. Therefore, he is a fool. It may however, turn out, that the fool will eventually get his reward — after *infinitely* many steps. In this sense, the strategy of the fool turns out to be successful and in our perception the fool becomes wise. We will apply such kind of strategy to prove the Fundamental Theorem of Chomp Game.

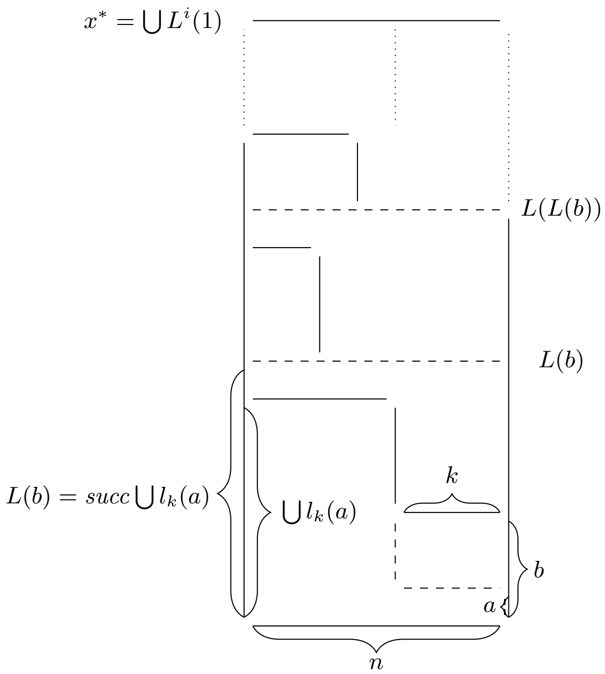
Let us consider a directly reachable configuration $d_k(\lambda, a)$ from *any* chocolate whose first dimension is n.

Observe that for a given $k < n$ and for a given a there is at most one λ such that $d_k(\lambda, a)$ is a losing configuration. Denote by $l_k(a)$ this ordinal number if it exists (and set it to 0 otherwise). Consider:

$$L(b) = succ\{l_k(a) \colon 0 \leq a < b, 0 \leq k < n\}$$

where $succX = X \cup \bigcup X$ is the smallest ordinal number strictly greater than all ordinal numbers in X. The idea is that for a given $l_k(a)$ position $n \times l_k(a)$ must be winning (because, we may decrease k-last numbers from $l_k(a)$ to a obtaining a losing position), therefore we may consider the next ordinal number $succl_k(a)$ for the second dimension of our losing configuration. Clearly, $L(b)$ is weakly increasing. One may also see that $L(1) = 2$ (because 1×1 is losing) and $L(2) = n + 1$ (by mirroring).

Now, let us define $x^* = \bigcup_{i \in \mathcal{N}} L^i(1)$ to be the smallest ordinal greater or equal than all $L^i(1)$, where L^i is the i-th fold composition of L with itself. The whole construction looks like on the following picture:



Now consider the game on chocolate $n \times x^*$. If the initial position is a losing position then the theorem is valid. So assume that it is a winning position and consider any losing position $d_k(x^*, a)$ reachable from it in a single move. We show that $k < n$ is not possible. We have that $a < x^*$ and by definition of x^* that: $a < L^i(1) \leq L^{i+1}(1)$ for some i. But by definition:

$$L^{i+1}(1) = L(L^i(1)) = succ\{l_k(a)\colon 0 \leq a < L^i(1), 0 \leq k < n\}$$

Therefore $x^* = l_k(a) < L^{i+1}(1)$, which leads to the contradiction since $L^{i+1}(1) \leq x^*$ by the definition of x^*. Therefore, $k = n$ and the position $\langle \underbrace{a, a, \ldots, a}_{n} \rangle$ is losing. But this is just the starting position in $n \times a$ chocolate game, which completes the proof.