Tutorial 7 20.11

1. How to find all P-invariants of a given net?

* P-invariant I has to satisfy equation $I \cdot N = \vec{0}$ where $N = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ t_1 & t_2 & \cdots & t_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ is a matrix consisting of all available transitions
* hence, to find all P-invariants, one has to solve the set of linear equations

2. Define a polynomial algorithm checking whether a given set of linear equations has a solution satisfying a given set of implications of form $x \geqslant 0 \Rightarrow y \geqslant 0$, where x, y are variables.

* a detailed solution is presented in the following video (by Piotr Hofman)

https://drive.google.com/file/d/11U54tA4LXrJoqlJswaMTf7a-bQXlmv1q/view?usp=sharing

3. Structural unboundedness for general Petri nets belongs to NP.

What condition should be satisfied by a net to be structurally unbounded?

Intuition: there exists a sequence of transitions that has a non-negative effect on all places and positive effect on at least one of them

To justify it, we can prove the following fact: the conditiones listed below are equivalent:

$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ t_1 & t_2 & \cdots & t_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

1) place p is structurally bounded,

2) there exists $y \geqslant \mathbb{1}_p$ s.t. $y^T N \leqslant \mathbf{0}$, where $\mathbb{1}_p \in \mathbb{Z}^{|P|}$, $\mathbb{1}_p[q] = 1$ if $q = p$ and 0 otherwise,

3) no $x \geqslant \mathbf{0}$ satisfies $Nx \geqslant \mathbb{1}_p$. $x \in \mathbb{N}^d$, $y \in \mathbb{N}^n$

Why can't we use coverability tree?

structurally
$\|$
$\forall$ configuration

$1) \Rightarrow 3)$

* we will prove $\neg 3) \Rightarrow \neg 1)$ (which is equivalent)
* let $x \in \mathbb{N}^d$ be a vector such that $Nx \geqslant \mathbb{1}_p$
* we can interpret x as a multiset of transitions and say that we fire x in a given net, meaning that we fire each transition the number of times it occures in x
* let M be the initial configuration big enough to fire x, then from M we obtain $M_1 > M$ on at least place p
* of course we can fire x in M_1 and iterate this process further, proving that place p is unbounded

$3) \Rightarrow 2)$

* to prove this implication, we use the following fact from the theory of dual programs

Theorem. Exactly one of the following equation systems has a solution:

Farkas' lemma

1) $Ax \geqslant b$

2) $\bar{y} \geqslant \mathbf{0}$
$\bar{y}^T A = \mathbf{0}$
$\bar{y}^T b > 0$

* we have to prove:

no $x \geqslant \mathbf{0}$ s.t. $Nx \geqslant \mathbb{1}_P$ $\Rightarrow$ $\exists\, y \geqslant \mathbb{1}_P$ s.t. $y^T N \leqslant \mathbf{0}$

* we observe the following

$$\begin{bmatrix} N \\ I \end{bmatrix} \cdot x \geqslant \underbrace{\begin{bmatrix} \mathbb{1}_P \\ \mathbf{0} \end{bmatrix}}_{=:\, b}$$

no solution

$\Rightarrow$

$$\bar{y} \geqslant \mathbf{0}$$
$$\bar{y}^T \cdot \begin{bmatrix} N \\ I \end{bmatrix} = \mathbf{0}$$
$$\bar{y}^T \cdot b > 0$$

has solution

* green box represents $Nx \geqslant \mathbb{1}_P$ and $x \geqslant \mathbf{0}$

* to analyze the blue box let us write

$\bar{y}^T = [\, y^T \,\vdots\, z^T \,]$

* it is easy to notice that $\bar{y} \in \mathbb{Q}^{|P|+|T|}$ solution

* first, it holds that

$$[y^T \vdots z^T] \cdot \begin{bmatrix} N \\ \hline I \end{bmatrix} = \mathbf{0} \iff \underbrace{y^T N + \underbrace{z^T}_{\geq \mathbf{0}} I}_{\geq \mathbf{0}} = \mathbf{0}$$

hence, $y^T N \leq \mathbf{0}$

* second, we have that $\bar{y} \geq \mathbf{0} \Rightarrow y \geq \mathbf{0}$ and

$$[y^T \vdots z^T] \cdot \begin{bmatrix} \mathbb{1}_p \\ \mathbf{0} \end{bmatrix} > 0 \iff y^T \cdot \mathbb{1}_p + z^T \cdot \mathbf{0} > 0$$

thus, $y^T \cdot \mathbb{1}_p > \mathbf{0} \Rightarrow y[p] > 0$

* however, $y \in \mathbb{Q}^{|P|}$ and hence we have to multiply all its coordinates by some number $K \in \mathbb{N}$ s.t. $Ky \in \mathbb{N}^{|P|}$ and $(Ky)[p] \geq 1$

2) $\Rightarrow$ 1)

* we assume that $\exists\, y \geq \mathbb{1}_p$ s.t. $y^T N \leq \mathbf{0}$
* look at initial conf. i and any reachable $z = i + Nx$
* we would like to prove $z[p] \leq C$ for some $C \in \mathbb{N}$ ← one for all z
* we can consider bounding $\overbrace{y[p]}^{\geq 1} \cdot z[p]$ instead
* $y[p] \cdot z[p] \leq y^T z = \underbrace{y^T i}_{:= m \geq 0} + \underbrace{y^T N}_{:= u \leq 0} x = m + u x \leq m$

both non-negative

$u \leq 0$, $x \geq 0$

* finally, we obtain $z[p] \leqslant \frac{m}{y[p]}$

Thus, to check structural unboundedness it is enough to check whether there exists a sequence of transitions that has a non-negative effect on all the places and positive effect for at least one of them.

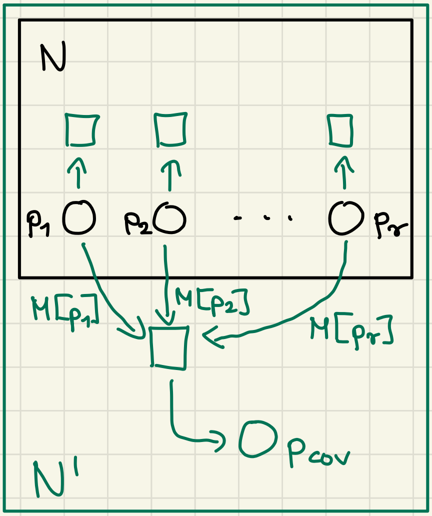
* we use analoguous algorithm as the last time (integer programming)
* this time instead of equalities we have a set of inequalities
* we have to guess a place that increases its token count after firing the sequence of transitions

* work on the details of this algorithm as an exercise

4. How to reduce coverability to reachability in general Petri nets?

from now on we assume that each initial net has r places

First solution

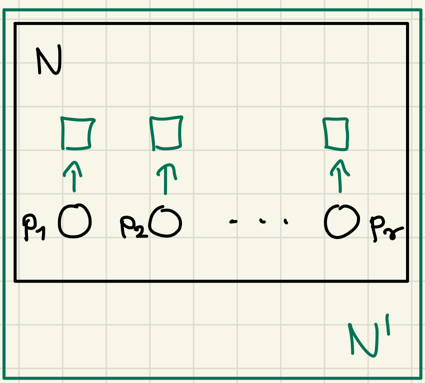


(N, M_0): can we cover M?

$$\underbrace{\left(M_0, \overset{\overset{p_{cov}}{\downarrow}}{0}\right)}_{=\,M_0'}$$

(N', M_0'): can we reach $(0, 0, \ldots, 0, 1)$?

Second solution



(N, M_0): can we cover M?

(N', M_0): can we reach M?

5. Prove that the submarking reachability problem (reaching a configuration with fixed numbers of tokens at a subset of net's places) is reducible to the zero reachability problem.

(P, T, F)

* given a net (N, M_0) and a submarking $M_{\tilde{P}}$ over a subset $\tilde{P} \subseteq P$, is there a reachable configuration M that agrees with $M_{\tilde{P}}$ over $\tilde{P}$?

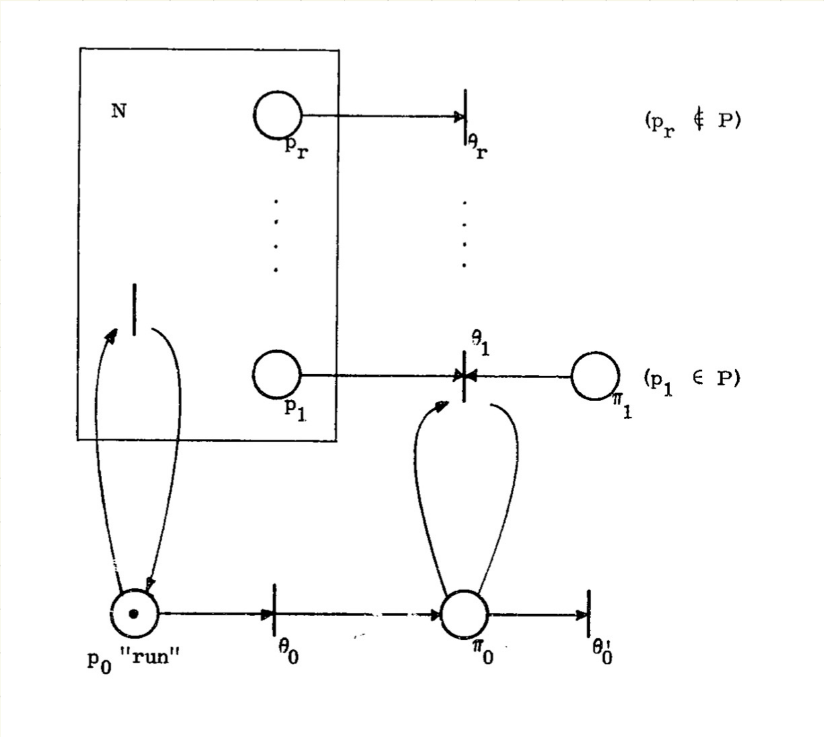
* we will answer this question by asking for zero configuration reachability in another net (obtained by modifying (N, M_0))

* first, we add a „run" place p_0 that contains one token and is connected with each transaction of the original net in both ways, i.e., for each $t \in T$: $p_0 \in {}^{\bullet}t$ and $p_0 \in t^{\bullet}$ (arc weight = 1)

* next, we add a transition θ_0 that takes one token from p_0, i.e., stops the normal run

of N and puts a token on the „check" place Π_0 allowing the firing of new transitions $\Theta_1, \Theta_2, \dots, \Theta_r$

* for $i \in P \setminus \tilde{P}$ transition Θ_i takes a token from p_i
* for $i \in \tilde{P}$ transition Θ_i takes one token from p_i and one from Π_i (Π_i initially has $M_{\tilde{P}}(p_i)$ tokens)
* finally, transition $\Theta_0^!$ that takes a token from Π_0



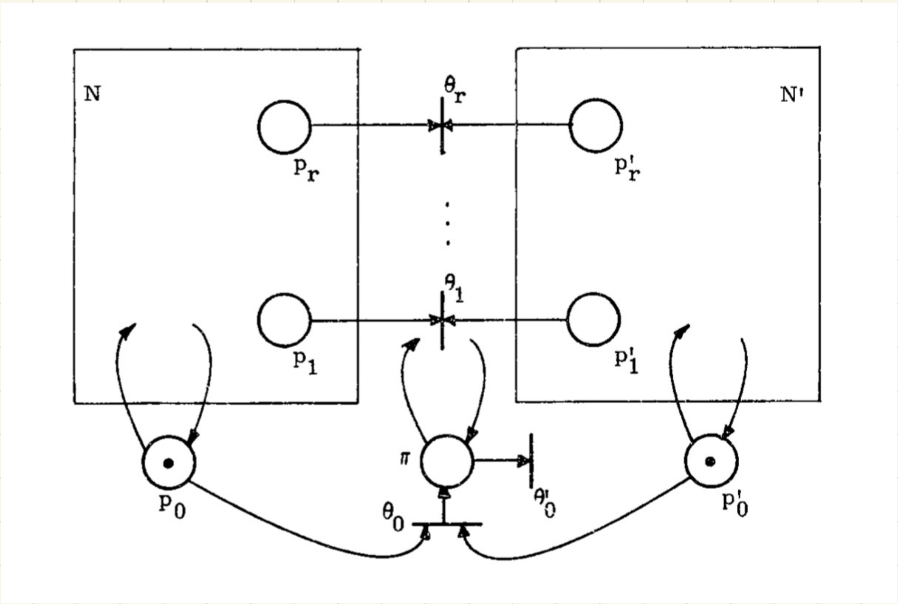
* it is obvious that zero configuration is reachable iff configuration $M_{\tilde{P}}$ over $\tilde{P}$ is reachable in N

A set $S \subseteq \mathbb{N}^r$ is RP-solvable iff the problem of deciding whether there exists a reachable configuration in S for a given net N with initial configuration M_0 is reducible to RP.

6. Every Reachability Set (a set of all configurations reachable in some net (N, M_0)) is RP-solvable.

* let $R_N(M_0) \subseteq \mathbb{N}^r$ be the Reachability Set of (N, M_0)
* our task is to show that for every other Petri net (N', M_0') of r places we can decide whether $R_N(M_0) \cap R_{N'}(M_0') \neq \emptyset$ using a blackbox for RP
* instead of reducing to RP, we can reduce to 2RP
* given N, N', we construct a new net N''
* for each of the initial nets, we add a new „run" place – p_0 and p_0', respectively – as we've done previously

* to stop the usual runs of N and N', we add a new transition θ_0 that takes a tokens from p_0 and p_0', and puts a token on a new place π
* then, we check whether configurations of N and N' agree using transitions $\theta_1, \theta_2, \ldots, \theta_r$



* it is easy to see that N'' can reach the zero configuration iff some configuration can be reached in both N and N'

1^D. Prove that reachability is reducible to non-liveness in general Petri nets.

homework (not obligatory)