Tutorial 6 14.11

1. Readhability for VASS with negative semantic belongs to NP.

<u>Brief theoretical introduction (hint)</u>:

Linear programming (LP)

a minimization / maximization problem for a linear function of n arguments: $x_1, x_2, \ldots, x_n$, where the arguments have to satisfy some conditions (linear equations or inequalities using x_is)

Example:

solvable in polynomial time

$$\max \quad x_1 + 2x_2$$

$$\text{s.t.} \quad x_2 \leqslant x_1 + 2$$

$$2x_1 + x_2 \leqslant 4$$

Integer programming (IP)

for each $i \in \{1, 2, \ldots, n\}$ we add a condition $x_i \in \mathbb{Z}$

this problem is NP-complete

Given: VASS (Q, T)

Question: $(p, u) \dashrightarrow^{*} (q, v)$?

does there exist a run that can drop below zero

Attempt 1

* denote transition vectors by $v_1, v_2, \dots, v_n$
* solve a set of linear equations

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = v - u$$

where x_i is the number of times transition t_i is to be fired (used)

Problem: we can obtain $x_i \notin \mathbb{Z}$ and $x_i < 0$

Attempt 2

and the goal of IP: $\min \sum_{i=1}^{n} x_i$

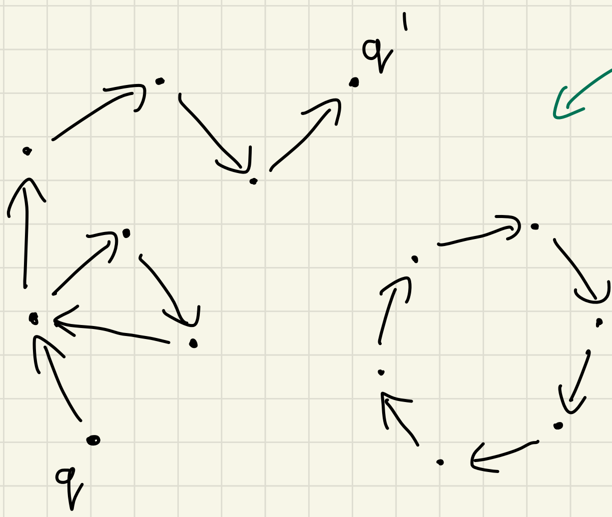
* add conditions: $x_i \in \mathbb{Z}$, $x_i \geq 0$ for each i

Problem: firing each transition t_i x_i times gives $v-u$ but the transitions may not form a run in VASS

Attempt 3

* add Kirchhoff's law conditions – for each state except q, q' the number of incoming transitions (sum of x_i) equals the number of outgoing ones
* for q there is one more outgoing transition and for q' – one more incoming

Problem: x_i can describe a path from q to q' and a disjoint cycle...

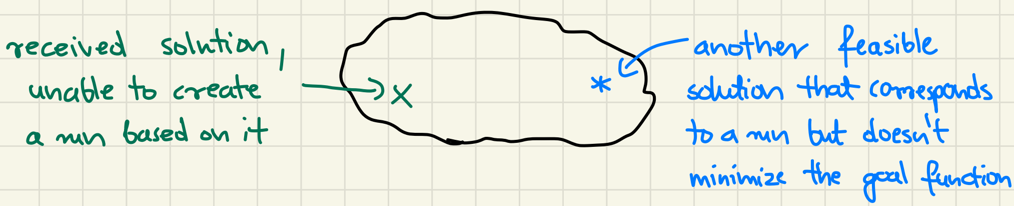


it satisfies our conditions

Attempt 4

* after obtaining x_is, define $S = \{t_i \in T \mid x_i > 0\}$ and check whether each $t_i \in S$ can be reached from q using transitions in S
* the condition above ensures that we can create the run given x_is (go from q using any available transition until reaching q' with the last transition incoming to q'; then, if we still have some spare transitions, we can create some cycles and glue them to the run $q \to^* q'$)
* it seems like it should be enough for S to be connected in order to create such a run

Problem: solution to our IP program may not satisfy the condition from attempt 4, but there can be some other run that does



Attempt 5 – final approach

* to ensure that every possible run is taken into account we can guess the transition set S' that we allow the use of and then solve IP with conditions

$$\forall_{i \in S'} \; x_i > 0$$

$$\forall_{i \in T \setminus S'} \; x_i = 0$$

2. Using the insights from the previous problem present a way to check condition Θ_1

Θ_1: for every $m \geq 1$ $(q, v) \dashrightarrow^* (q', v')$ using every transition $\geq m$ times

* if there exists a run r' for some selected $m' > 0$ using every transition t_i $x_i \geq m'$ times, then there also exists a run r'' that uses each transition at least $m'' = \max(x_i) + 1$ times (we use notation $\# t_i \rightsquigarrow y_i$ for r'')

* both r' and r'' satisfy the Kirchhoff's law, hence so does $r'' - r'$ which uses each transition t_i at least $y_i - x_i > 1$ times
* thus, $r'' - r'$ is a cycle that has no side effect on a given configuration
* the solution idea:
  - define two sets of variables : $x_1, x_2, \ldots, x_n$ and $\Delta_1, \Delta_2, \ldots, \Delta_n$ such that both sets satisfy Kirchhoff's law (first with a twist for q and q')
  - $x_i > 0$ (for a subset of transitions - as discussed before), firing all t_i s x_i times results in $v' - v$ change on the configuration
  - $\Delta_i > 0$ for all transitions and firing t_i s don't change the configuration

3. Prove that the following implication holds:

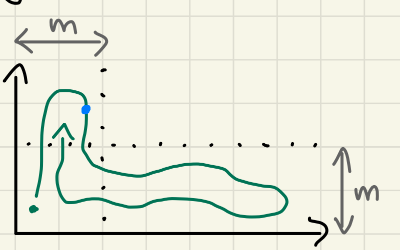
$$\left.\begin{array}{l}\exists m.\ \text{every configuration reachable from} \\ \quad (q, v) \text{ has some coordinate} < m\end{array}\right] \quad (1)$$

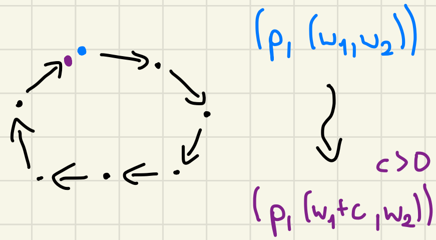
$$\Downarrow$$

$$\left.\begin{array}{l}\exists m.\ \text{every run from } (q, v), \text{ on some} \\ \quad \text{coordinate, is always} < m\end{array}\right] \quad (2)$$

* if (1) holds, we can set m to be the greatest integer appearing in the coverability tree (it might take a lot of time to find m but we can afford it; also, for VASSes we have to remember the state we are in)
* now, we show that such an m satisfies (2)
* first, assume that the VASS is 2-dimensional and assume that there exists a run r such that r goes through points (v_1, v_2) with $v_1 \geq m$ and (u_1, u_2) with $u_2 \geq m$

* it means that we had to reach $(p, (\omega, \_))$ in the coverability tree $\Rightarrow$ there exists a sequence σ of transitions that starts and ends in p, increases the first coordinate and doesn't change the second one
* we can fire σ as many times as it is required to increase the first coordinate by $\geq m$, i.e., we extend r in such a way



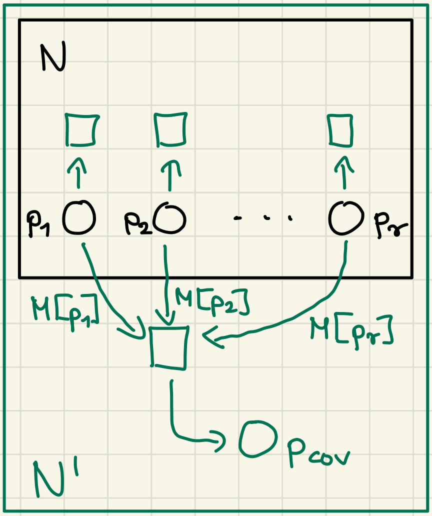


* it gives us a new run r' that instead of (u_1, u_2) reaches $(\overbrace{u_1+k}^{\geq m}, \overset{m}{\underset{\wedge}{u_2}})$ with $k \geq m$
* this configuration doesn't satisfy (1) – contradiction
* for higher dimentions the reasoning is similar (the only problem: going from $(\omega, x_1, \dots, x_d)$ to $(\omega, \omega, y_2, \dots, y_d)$ might decrease first coordinate)

4. How to reduce coverability to reachability in general Petri nets?

from now on we assume that each initial net has r places

First solution



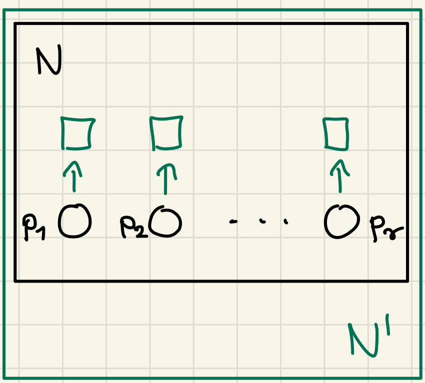
(N, M_0): can we cover M?

$\Downarrow$

$M_0' = (M_0, \overset{p_{cov}}{0})$

(N', M_0'): can we reach $(0, 0, \dots, 0, 1)$?

Second solution



(N, M_0): can we cover M?

$\Downarrow$

(N', M_0): can we reach M?

5. Show reductions between the following problems:

1) reachability problem: $M \in \mathbb{N}^r$, is M reachable? (RP)

2) submarking reachability problem: $\tilde{P}$ – subset of places, $M_{\tilde{P}}$ – configuration over $\tilde{P}$, does there exist a reachable configuration M such that over the places of $\tilde{P}$ it agrees with $M_{\tilde{P}}$? (SRP)

3) zero reachability problem: is $\mathbf{0}$ reachable? (ZRP)

4) single-place zero reachability problem: given a place p, does there exist a reachable configuration M, s.t. $M(p) = 0$? (SPZRP)

The idea is to show the following reductions:

$$\begin{tikzcd}
\text{SRP} \arrow[d] \arrow[dr, bend left] & \\
\text{RP} \arrow[d] & \text{SPZRP} \arrow[dl, Rightarrow, bend left] \\
\text{ZRP} \arrow[uu, Rightarrow, bend left] &
\end{tikzcd}$$

„Y is reducible to X"

notation: $X \rightarrow Y$, $X \Rightarrow Y$

know how to solve this (X)

can solve this (Y)

$\longrightarrow$ easy reduction

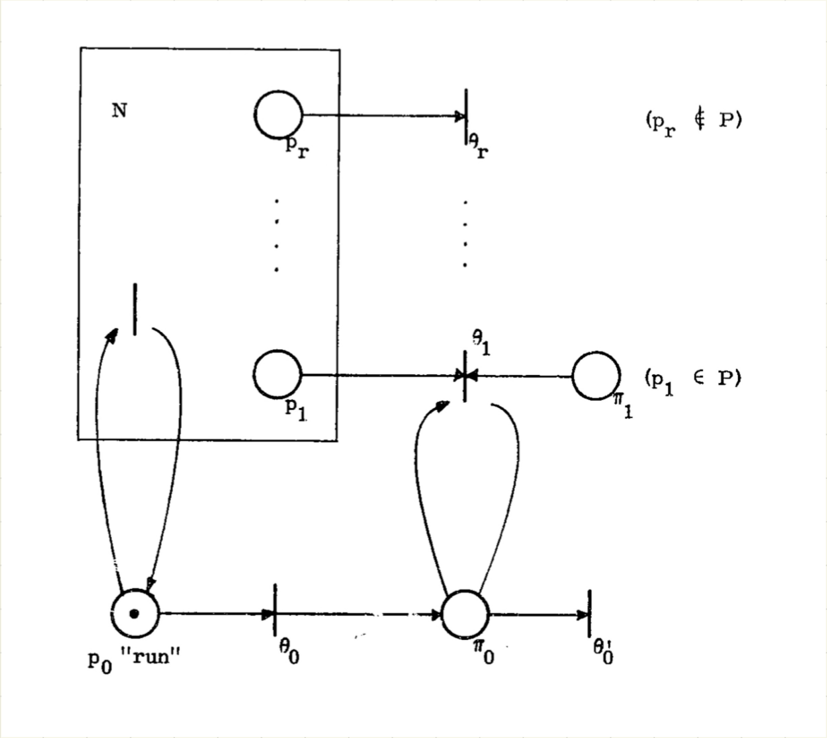
$\Longrightarrow$ reduction to present

2RP $\Rightarrow$ SRP:

* given a net (N, M_0) ($N = (P, T, F)$) and a submarking $M_{\tilde{P}}$ over a subset $\tilde{P} \subseteq P$, is there a reachable configuration M that agrees with $M_{\tilde{P}}$ over $\tilde{P}$?
* we will answer this question by asking for zero configuration reachability in another net (obtained by modifying (N, M_0))
* first, we add a „run" place p_0 that contains one token and is connected with each transaction of the original net in both ways, i.e., for each $t \in T$: $p_0 \in {}^\bullet t$ and $p_0 \in t^\bullet$ (arc weight = 1)
* next, we add a transition Θ_0 that takes one token from p_0, i.e., stops the normal run of N and puts a token on the „check" place π_0 – this token allows the firing of new transitions $\Theta_1, \Theta_2, \dots, \Theta_r$
* for $i \in P \setminus \tilde{P}$ transition Θ_i takes a token from p_i

* for $i \in \tilde{P}$ transition θ_i takes one token from p_i and one from π_i, π_i initially having $M_{\tilde{P}}(p_i)$ tokens

* finally, we have a transition θ_0' that takes one token from π_0

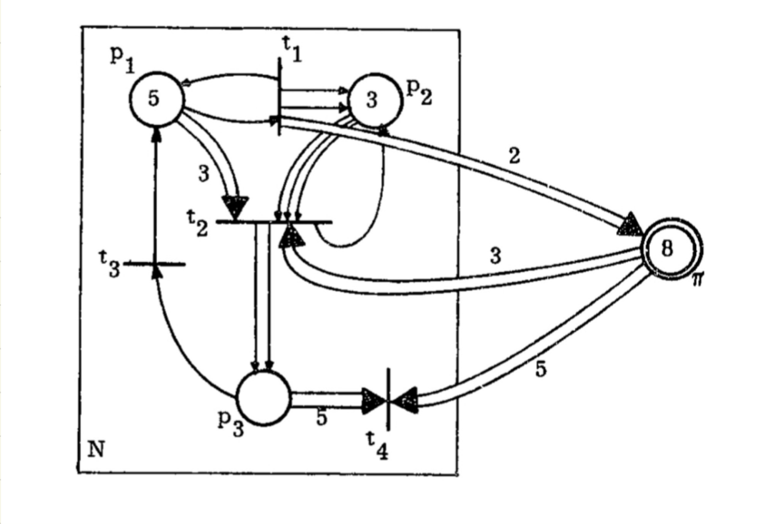


* it is obvious that zero configuration is reachable iff configuration $M_{\tilde{P}}$ over $\tilde{P}$ is reachable in N

SP2RP $\Rightarrow$ 2RP:

* now, we can use the blackbox for answering single-place zero reachability problem and have to answer the zero reachability problem in N
* the idea is to have a new place π s.t. at all times π contains as many tokens as there are in all places of N, i.e., at every configuration M in the new net N': $M(\pi) = \sum_{i=1}^{r} M(p_i)$
* to obtain it, we connect π with all transitions of N with incoming and outgoing arcs – the weight of incoming arc is the sum of incoming arcs' weights in N and analogously for outgoing
* initially, π contains the total number of tokens in configuration M_0 for N
* obviously, $t_j \in T$ is fireable in N iff t_j is fireable in N'
* answer for 2RP in N = answer for SP2RP in N'

* obtaining N' from N – example



Source of the net modification figures:

Michel Hack. Decidability Questions for Petri Nets

6. Structural unboundedness for general Petri nets belongs to NP.

homework (not obligatory)