Stochastic games and their complexities

public defense

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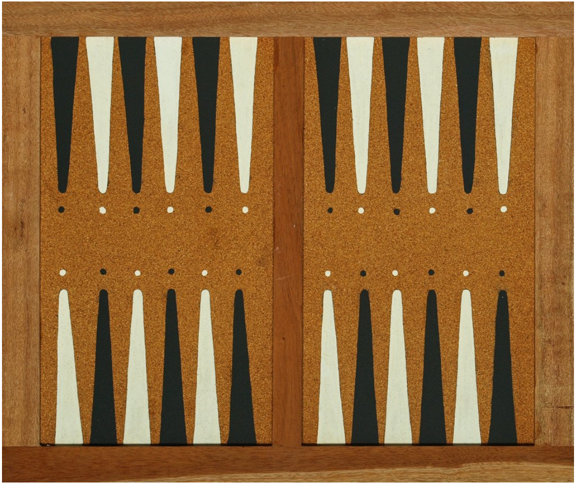
Stochastic games

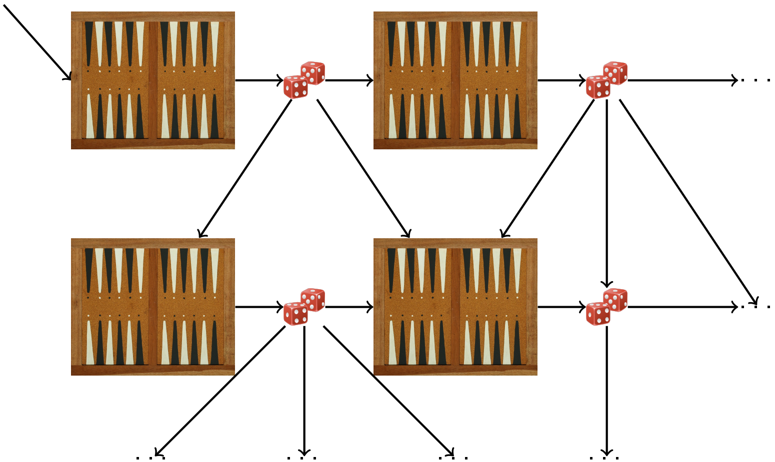
Games with probabilistic transitions



backgammon board

Stochastic games

Games with probabilistic transitions

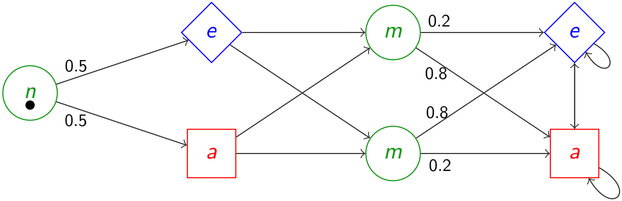


graph of configurations

Formal model

Games on graphs

- move a single token along the lines of a graph



- decide the outcome based on the history of the token

a word over the labels of the vertices

Stochastic branching games

I study

- two-player zero-sum games

  e.g. chess

- of infinite duration

  e.g. parity games

- with randomness

  e.g. simple stochastic games

- and a form of concurrency.

  i.e. stochastic branching games by Mio

Strong model: subsume many turn-based games

extension of Gale-Stewart games

and some concurrent games.

can encode Blackwell games

Branching game

Game $G = \langle \mathtt{B}, \Phi \rangle$ is played by two players (Eve and Adam) and consists of

- a *board* $\mathtt{B}$

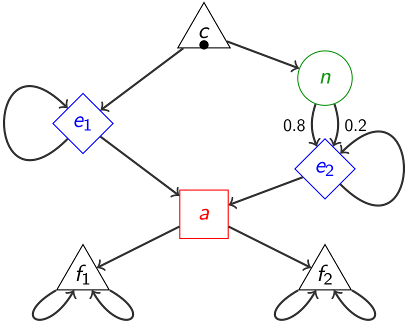
  the graph where we move the tokens

- and an *objective* $\Phi\colon \mathrm{plays}(\mathtt{B}) \to [0, 1]$.

  ... that can be seen as a *rulebook* ...

Branching games, an example

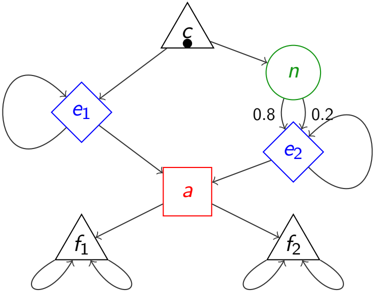
Board – (unfolding of a) binary graph, consisting of Adam's, Eve's, Nature's and branching vertices.



The black token marks the initial vertex.

Branching play

Board



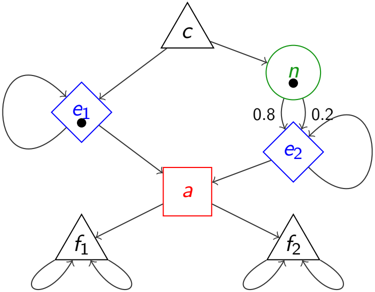
Play

c

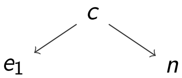
preplay – a labelled tree with tokens

Branching play

Board



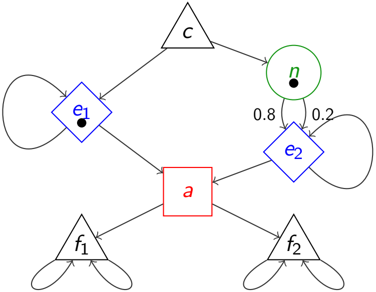
Play



preplay – a labelled tree with tokens

Branching play

Board



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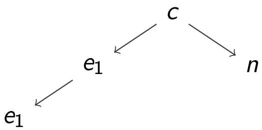
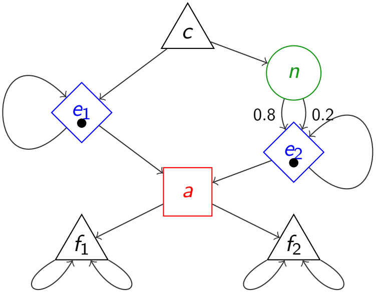


preplay – a labelled tree with tokens

Branching play

Board



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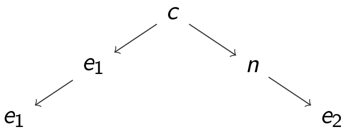

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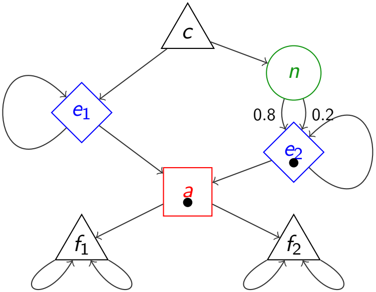
Branching play

Board



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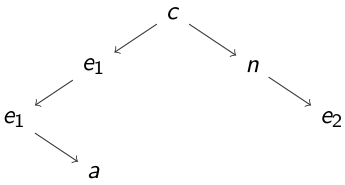

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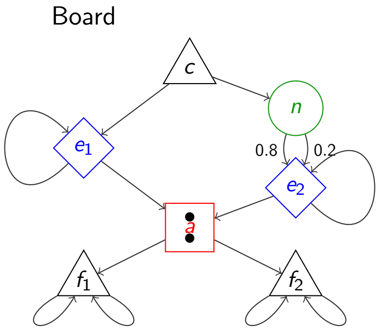
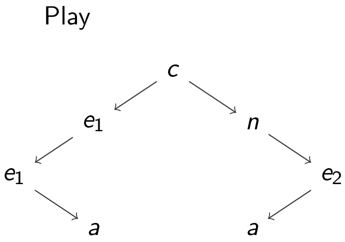
Branching play





preplay – a labelled tree with tokens

Branching play

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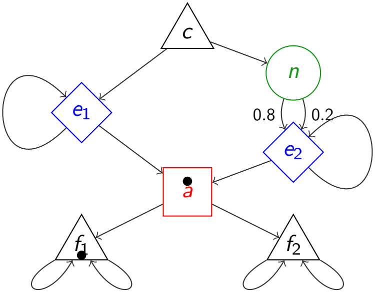

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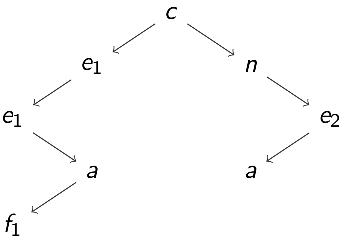
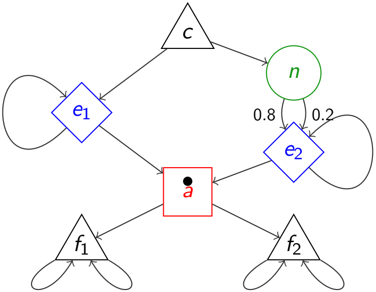


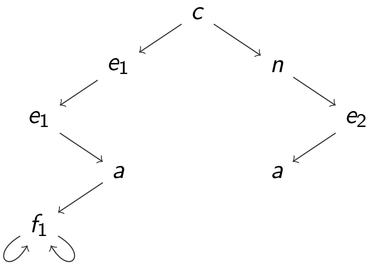
preplay – a labelled tree with tokens

Branching play

Board



Play



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Branching play

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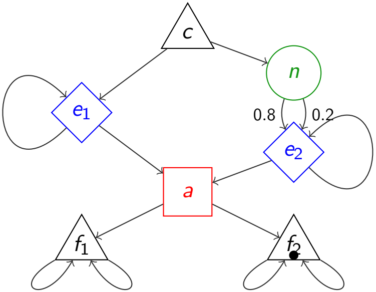

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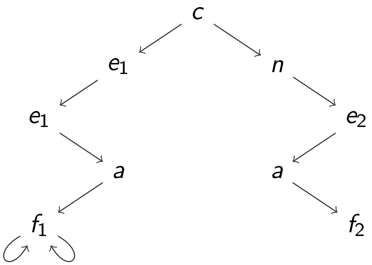

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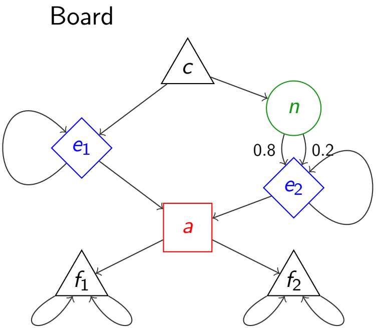
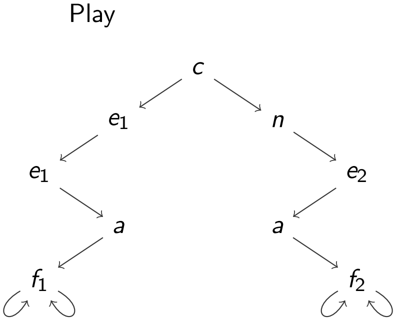
Branching play

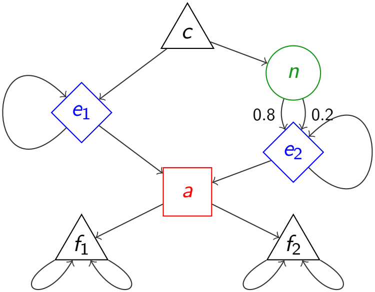




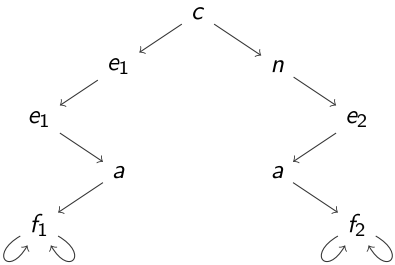
preplay – a labelled tree with tokens

Branching play

Board



Play



play – an infinite labelled tree

Games and measure

Fix a game G and a profile of strategies $\langle \sigma, \pi \rangle$;

$\sigma, \pi \colon \{\mathtt{L}, \mathtt{R}\}^* \to \{\mathtt{L}, \mathtt{R}\}$

the *outcome* generates a measure $\mu^{\sigma,\pi}$ defined on a set $\mathcal{T}_\Gamma$;

$\mathcal{T}_\Gamma$ – a set of trees labelled with some finite set Γ.

the *objective* is a function $\Phi : \mathcal{T}_\Gamma \to [0, 1]$

Φ is called the pay-off function
Eve wants to maximise the pay-off,
Adam to minimise it.

the *value* $val(\sigma, \pi)$ of a profile is the expected pay-off $\mathbb{E}\Phi$

wrt. the measure $\mu^{\sigma,\pi}$

Two questions

Problem (determinacy)

Given a game G, is the game determined?

pure and mixed determinacy

Problem (game value)

Given a game G, what is the value of the game G?

six possible values

Game values and determinacy

Let G be a branching game,

- Eve's value:
  $val^{XE} \overset{\text{def}}{=} \sup_{\sigma \in \Sigma^{XE}} \inf_{\pi \in \Sigma^{A}} val(\sigma, \pi)$
- Adam's value:
  $val^{XA} \overset{\text{def}}{=} \inf_{\pi \in \Sigma^{XA}} \sup_{\sigma \in \Sigma^{E}} val(\sigma, \pi)$

X ranges over pure (ε), behavioural (B), and mixed (M) strategies

Strategies

- pure $\sigma \colon \{\mathtt{L}, \mathtt{R}\}^* \to \{\mathtt{L}, \mathtt{R}\}$,
- mixed $\sigma_m \in Dist(\sigma \colon \{\mathtt{L}, \mathtt{R}\}^* \to \{\mathtt{L}, \mathtt{R}\})$.

$\Sigma^{E}_{\mathtt{B}} \subsetneq \Sigma^{BE}_{\mathtt{B}} \subsetneq \Sigma^{ME}_{\mathtt{B}}$

$\Sigma^{A}_{\mathtt{B}} \subsetneq \Sigma^{BA}_{\mathtt{B}} \subsetneq \Sigma^{MA}_{\mathtt{B}}$

Determinacy

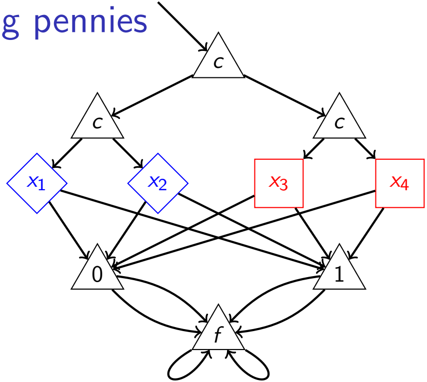
Holds

$$val^{A} \geq val^{BA} \geq val^{MA} \geq val^{ME} \geq val^{BE} \geq val^{E}$$

Game G is *determined*

- *under pure strategies* if $val^{A} = val^{E}$,
- *under behavioural strategies* if $val^{BA} = val^{BE}$,
- *under mixed strategies* if $val^{MA} = val^{ME}$.

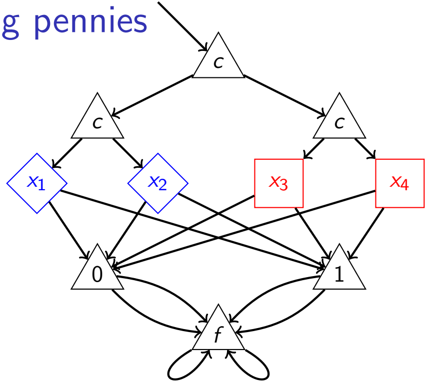
Double matching pennies



$$L \stackrel{\text{def}}{=} \{ t \in \text{plays}(\mathtt{B}) \mid (x_3(t){=}x_4(t)) \implies (x_1(t){=}x_2(t){=}x_3(t){=}x_4(t)) \}$$

example by Mio

Double matching pennies



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example by Mio

Game values:

val^A	val^{BA}	val^{MA}	val^{ME}	val^{BE}	val^E
1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0

Regular branching games

Fix a game G and a set of trees L

if $\Phi : \mathcal{T}_\Gamma \to [0,1]$ of form

$$\Phi(t) = \begin{cases} 1 & \text{if } t \in L \\ 0 & \text{otherwise} \end{cases}$$

then L is called the winning set of G.

L regular $\implies$ G regular branching game.

regular means recoginsable by a finite automaton on trees

Regular pure branching games

Regular pure two-player games

no *Nature*'s vertices

Main results:

- games may not be determined, not even under mixed strategies;

  we provide an example

- pure values are computable;

  winning strategies are enough

- the determinacy under pure strategies can be decided.

  we can compute the values

Regular pure branching games

Theorem (indeterminacy)

There exists a regular finite branching pure game G such that

val^A	val^{BA}	val^{MA}	val^{ME}	val^{BE}	val^E
1	1	1	0	0	0

the winning set is a difference of two open sets

Theorem (computing the value)

Let G be a regular finite pure branching game. Then the set of winning strategies S_E (resp., S_A) of Eve (resp., Adam) is regular and effectively computable.

$val^E = [S_E \neq \emptyset], val^A = [S_A \neq \emptyset]$

Regular stochastic branching games

Regular stochastic branching games

Main results:

- determined under mixed strategies, if the winning set is topologically closed;

  recall: difference of two closed sets can be indeterminate

- the values are **not** computable, not even for one-player games;

  nor even mixed values of two-player pure games

Regular stochastic two-player games

Theorem (determinacy)

Let G be a regular stochastic branching game with a closed winning set. Then G is determined under mixed strategies.

pay-off function is semicontinous $\implies$ apply Glicksberg's minimax theorem

Theorem (computing the value)

There is no algorithm that given a regular finitary branching game G computes the value of the game.

reduction from probabilistic automata

Measures

Regular zero-player stochastic games

Fix strategies σ, π and a board $\mathtt{B}$, then $\mu^{\sigma,\pi} \colon 2^{\mathcal{T}_\Gamma} \rightharpoonup [0,1]$ defined as

$$\mu^{\sigma,\pi}(L) = \mathit{val}_{\langle \mathtt{B}, L\rangle}(\sigma, \pi)$$

is a Borel measure.

We focus on the uniform measure on infinite binary trees μ^*

for any position $u \in \{\mathtt{L}, \mathtt{R}\}^*$ and any letter $a \in \Gamma$

$$\mu^*(\{t \in \mathcal{T}_\Gamma \mid t(u) = a\}) = |\Gamma|^{-1}$$

Measures

regular means recoginsable by a finite automaton

Main result:

the uniform measure of a regular set of trees L is computable, if L is defined by

- a first-order formula with no descendant,
- or a conjunctive query.

both results utilise a form of locality

Measures

Theorem (FO case)

If φ is a first-order formula not using descendant, then $\mu^*(L(\varphi))$ is rational and computable in three-fold exponential space.

Theorem (CQs case)

If φ is a Boolean combination of conjunctive queries, then $\mu^*(L(\varphi))$ is rational and computable in exponential space.

we reduce $L(\varphi)$ to a *clopen* set having the same measure

Plantation game

Plantation game

Goals:

- practical part of the thesis;
- presentation of potential use of discrete stochastic games;
- beachhead for the future, more involved, cooperation.

Problem to solve:

- given a history of a fruit plantation

  a time series of measurements

- produce a tool to predict and control the level of infestation.

  solution: stochastic games

Approach

Utilise a standard framework:

1. convert raw data into series with finite number of values

   clustering

2. create a model simulating the changes

   partially observable Markov decision process

3. create a game-based reactive model

   a Gale-Stewart-like game with hidden states

Result:

simple and procedural framework creating reactive models for given time series of observations

Conclusions

Determinacy:

- mixed determinacy, if the winning set is closed;
- mixed determinacy fails at the second level of Borel hierarchy;

Values:

- pure values of pure regular branching games are computable;
- values of stochastic regular branching games are not.

Measures:

- uniform measure can be computed for some restricted first-order formulae.

Applications:

- a description of a roboust framework and a proposition of a possible implementation.