The stubborn problem is stubborn no more

a polynomial algorithm for 3-compatible colouring and the stubborn list partition problem

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Problem definition

3-Compatible colouring

Input: A complete undirected graph $G = (V, E)$ and a function $\mathcal{C} : E \to \{\mathcal{R}, \mathcal{G}, \mathcal{B}\}$

Question: Does there exist a function $\phi : V \to \{\mathcal{R}, \mathcal{G}, \mathcal{B}\}$ such that for each edge $uv \in E$ either $\phi(u) \neq \mathcal{C}(uv)$ or $\phi(v) \neq \mathcal{C}(uv)$

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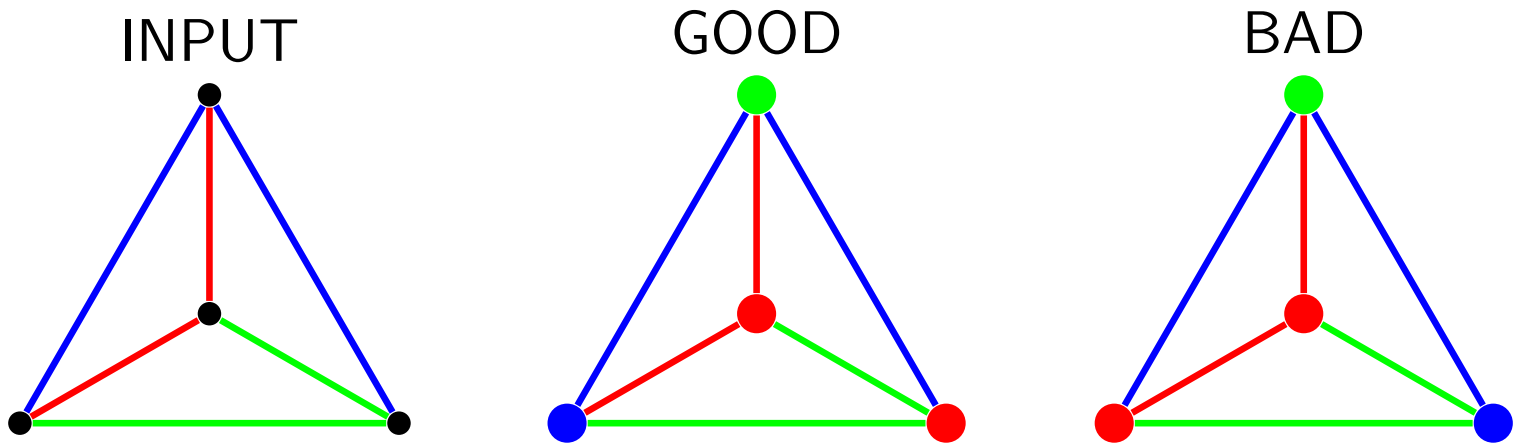

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- 3-Compatible colouring was known to be as hard as the Stubborn problem, an irritating case resisting classification of List Matrix Partition problems by Cameron et al. (SODA 2004).

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- Developing a polynomial algorithm for 3-Compatible colouring became a nurturing open problem that inspired a lot of research.
- **Our result:** an $O(n^7)$ algorithm.

Warmup: two colours

2-Compatible colouring

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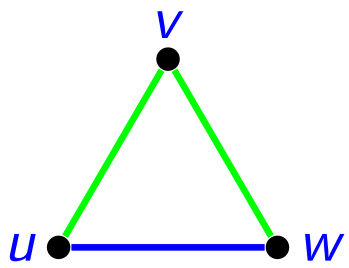
Two colours: solution

- **Observation:** If $\mathcal{C}(uv) = \mathcal{C}(vw) \neq \mathcal{C}(uw)$, then v has determined colour.



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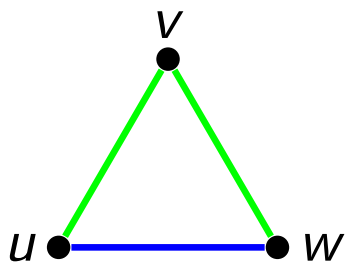

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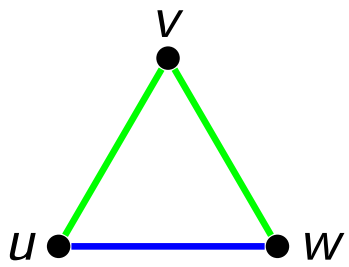

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  - Until the graph becomes a monochromatic clique, find such a triangle, determine the colour of the vertex and propagate this information by deleting vertices with already decided colours.

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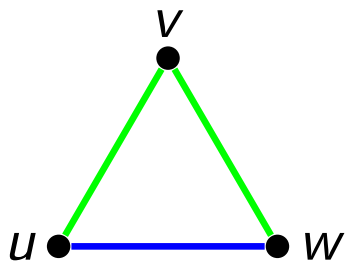

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  - Until the graph becomes a monochromatic clique, find such a triangle, determine the colour of the vertex and propagate this information by deleting vertices with already decided colours.
  - Then we have $m + 1$ solutions: all the vertices of the second colour, or just one of the first.

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- **Goal:** we have (some) feasible colouring of a subclique (denoted $\mathrm{previous}$) and we seek a new colouring with one more vertex.
- **Note:** In the new colouring all the vertices can be repainted; the previous colouring is just a 'hint'.

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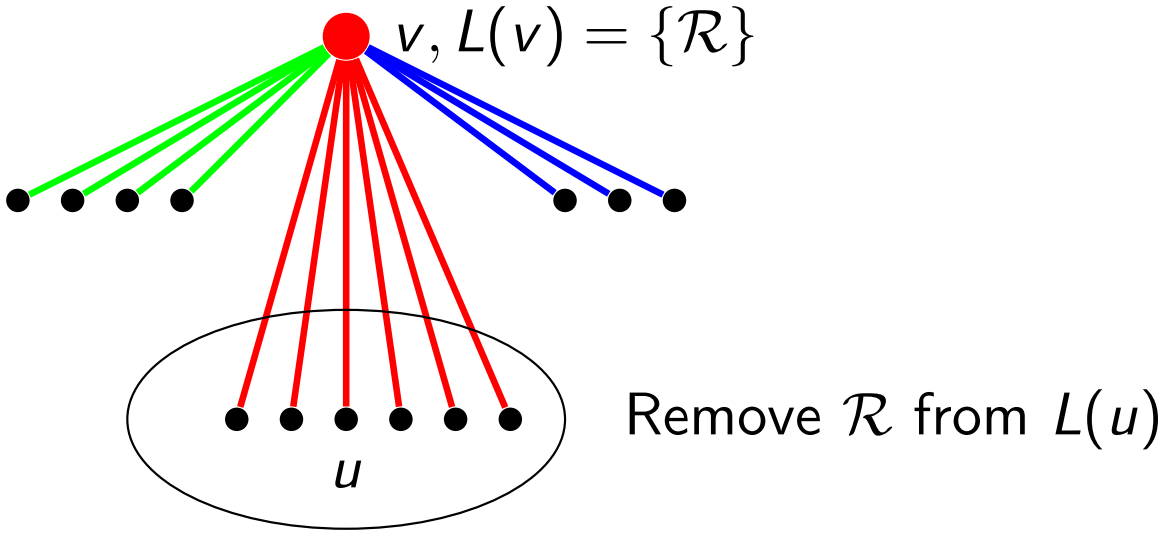
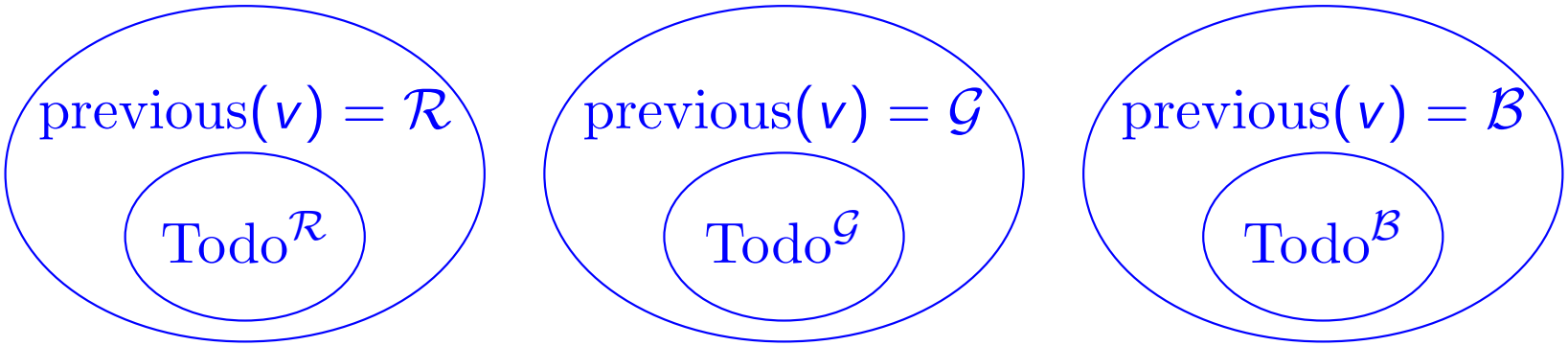


Structure of the problem

We define subsets of vertices according to their lists of available colours and the previous colour.

$$\text{Todo}^x = \{v : \text{previous}(v) = x, |L(v)| = 2, x \notin L(v)\}$$



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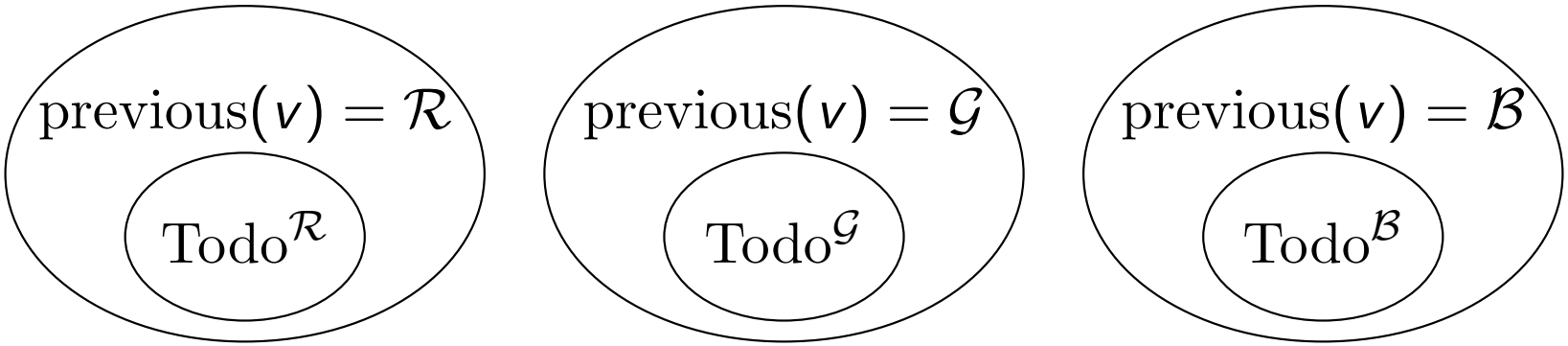

Intuition: Todo^x are the vertices that need to be repainted.

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Observation 2

In $E(\mathrm{Todo}^x)$ there are no edges of colour x, hence $G[\mathrm{Todo}^x]$ is an instance of 2-Compatible colouring.

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- And now the magic begins.
- We do simple branching: take some nonempty Todo^x and branch into $|\mathrm{Todo}^x| + 1$ possibilities.

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- Let ϕ, ϕ' be two different 2-compatible colourings on Todo^x in those branches.
- Let v_2 be a vertex such that $\phi(v_2) \neq \phi'(v_2)$.
- Since $\mathcal{C}(v_1 v_2) \neq x$, w.l.o.g. we may assume $\mathcal{C}(v_1 v_2) = \phi(v_2)$, hence v_1 is not free in the branch corresponding to ϕ, a contradiction.

Time analysis

- We bound the number of leaves of the branch tree by $O(n^3)$ using a potential function

$$\pi = (\mathrm{Free}^{\mathcal{R}} + \mathrm{Todo}^{\mathcal{R}} + 1)(\mathrm{Free}^{\mathcal{G}} + \mathrm{Todo}^{\mathcal{G}} + 1)(\mathrm{Free}^{\mathcal{B}} + \mathrm{Todo}^{\mathcal{B}} + 1).$$

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- Due to the Key Lemma, in the branching step the sum of the potentials in the branches is at most the prior potential.
- The time complexity is $O(n^7)$ in total.

Thank you for your attention

Questions?