LP branching, part II: Vertex Cover

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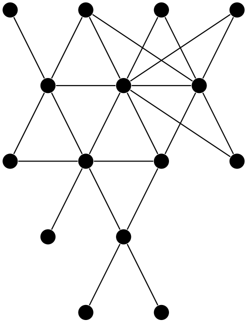
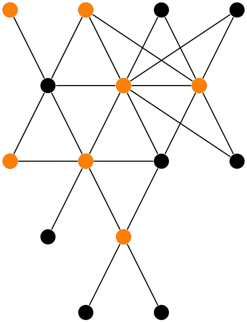
Finse, March 2014

Vertex Cover





Find minimum X
with $G \setminus X$ edgeless.

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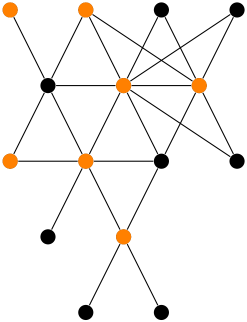

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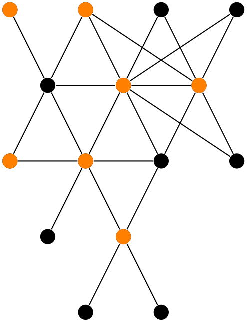

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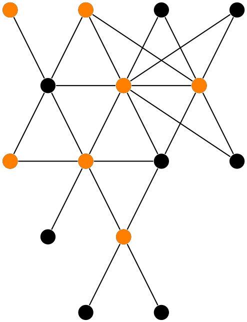
$\mathcal{O}^*(2^k)$ running time

$\mathcal{O}^*(1.2738^k)$, Chen, Kanj, Xia



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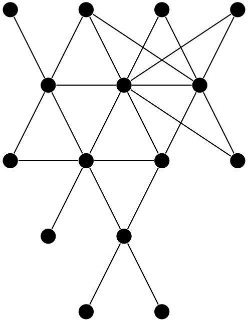
The size of the solution is often big. Better parameter?

Vertex Cover is a special case of Multiway Cut



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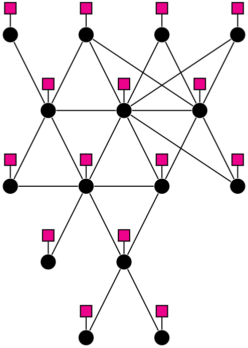

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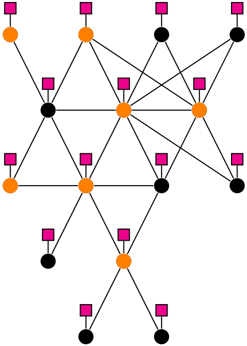

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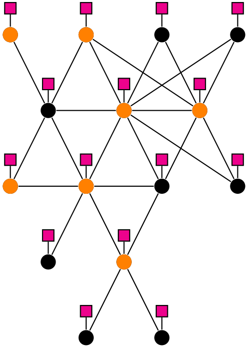

Multiway Cut LP:

$$\min \sum_{v \in V(G)} x_v$$

$$\text{s.t.} \sum_{v \in V(P)} x_v \geq 1 \quad \forall_{s,t \in T, s \neq t} \forall_{s-t \text{ path } P}$$

$$x_v \geq 0 \quad \forall_{v \in V(G)}$$

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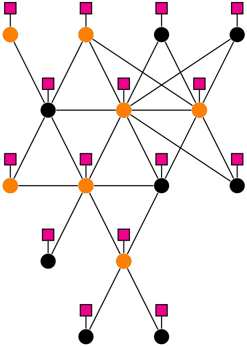

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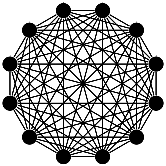
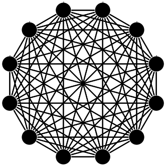
Our goal: analyse Vertex Cover LP





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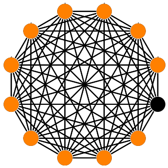
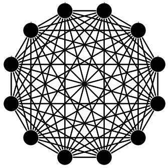


cost: $n-1$



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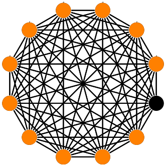

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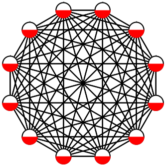

cost: $n/2$

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Vertex Cover LP: vertex classes

$$x_v = 0 \iff v \in V_0$$

$$0 < x_v < \tfrac{1}{2} \iff v \in V_\varepsilon$$

$$V_{<\frac{1}{2}} = V_0 \cup V_\varepsilon$$

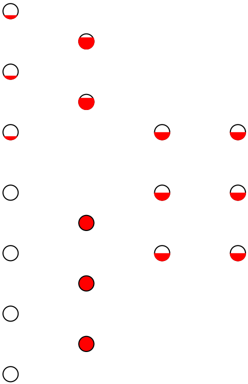
$$x_v = \tfrac{1}{2} \iff v \in V_{\frac{1}{2}}$$

$$\tfrac{1}{2} < x_v < 1 \iff v \in V_{1-\varepsilon}$$

$$x_v = 1 \iff v \in V_1$$

$$V_{>\frac{1}{2}} = V_1 \cup V_{1-\varepsilon}$$

Vertex Cover LP: analysis



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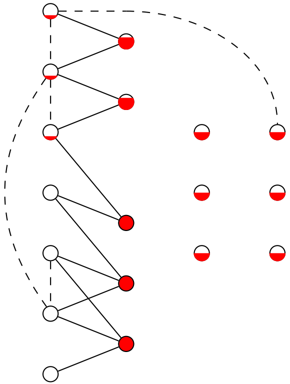

1. $V_{<\frac{1}{2}}$ independent, $E(V_{<\frac{1}{2}}, V_{\frac{1}{2}}) = \emptyset$,
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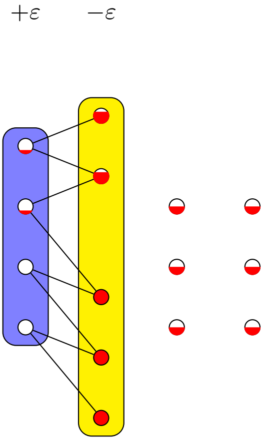

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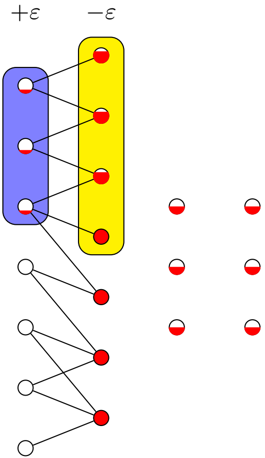

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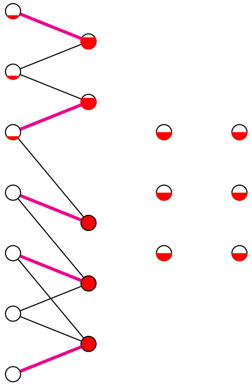

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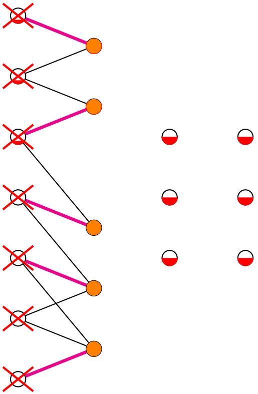

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5. Greedily take $V_{>\frac{1}{2}}$ and discard $V_{<\frac{1}{2}}$.

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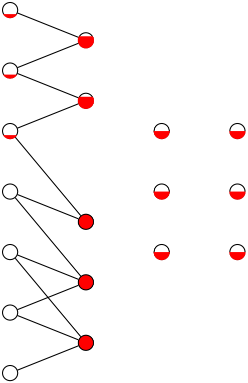

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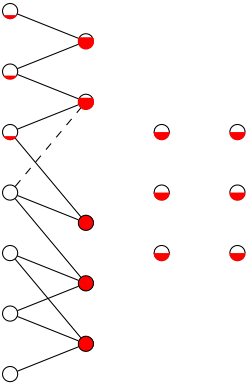

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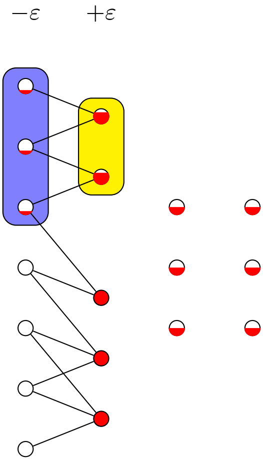

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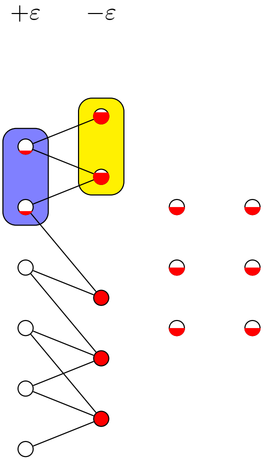

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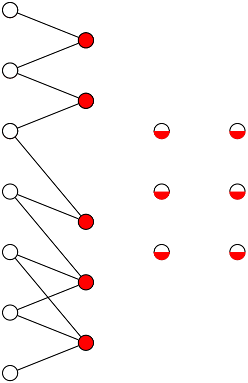

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4. conclusion: there exists LP optimum with
   $V = V_0 \cup V_{\frac{1}{2}} \cup V_1$.

Vertex Cover LP: summary of analysis

1. There exists LP optimum with $V = V_0 \cup V_{\frac{1}{2}} \cup V_1$.
    - Half-integrality
2. If $V_{\frac{1}{2}} \neq V$, then we can reduce something.
    - Take V_1 into solution, and discard V_0.

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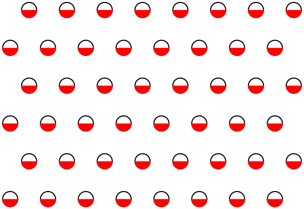

unique LP optimum

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Pick any vertex v.

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LP value goes up by $\frac{1}{2}$.

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Theorem (Cygan, P., Pilipczuk, Wojtaszczyk, 2011)

Vertex Cover *can be solved in* $\mathcal{O}^*(4^{k-LP})$ *time.*

Vertex Cover LP branching worst case

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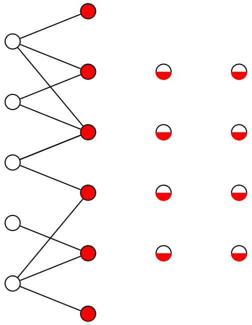

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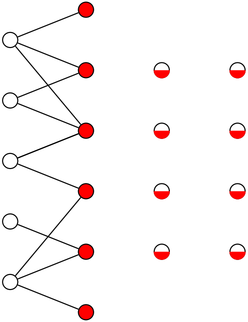

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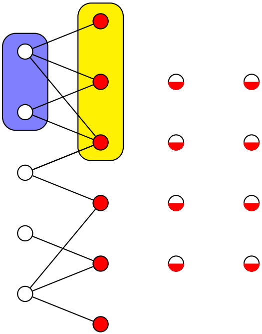

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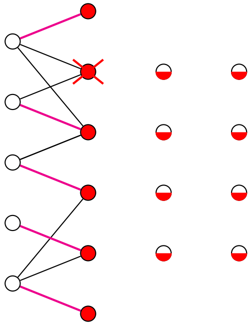

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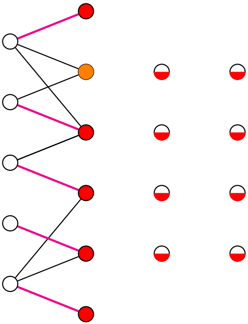

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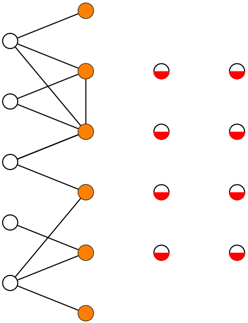

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- If $E(G[V_1]) \neq \emptyset$, then take V_1 greedily into solution.

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- Worst case: if forcing $x_v = 0$ increased LP optimum only by $\frac{1}{2}$.
- Forcing $x_v = 0$ is essentially:
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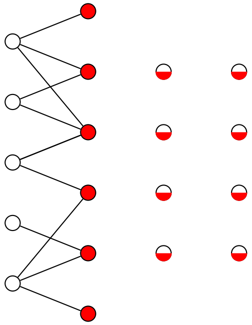

- $\forall_{X \subseteq V_0} |N(X)| > |X|$, otherwise $V_0 = X$, $V_1 = N(X) \Rightarrow$ LP optimum.
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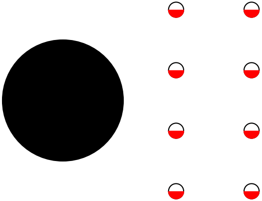

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Force $x_v = 1$ in the LP.

LP value goes up by $\frac{1}{2}$.

$(k - LP)$ goes down by $\frac{1}{2}$.

Branch 2:

Forbid v to be in the solution.

Force $x_v = 0$ in the LP.

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Vertex Cover *can be solved in* $\mathcal{O}^*(2.619^{k-LP})$ *time.*

Corollaries

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The following problems can be solved in $\mathcal{O}^*(2.619^p)$ time.

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- VERTEX COVER, *parameterized by $p = k - \mu$,*
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Open problem: solve VERTEX COVER in $\mathcal{O}^*(2^{k-LP})$ time.