# Computing treedepth faster than $2^n$

Fedor V. Fomin, Archontia C. Giannopoulou, Michał Pilipczuk

Department of Informatics, University of Bergen, Norway

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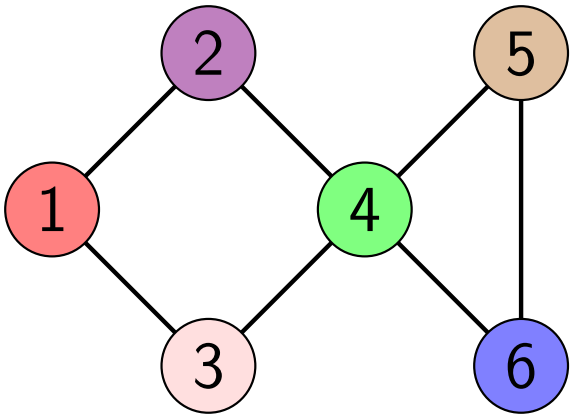
  - We also say that $F$ is a *treedepth decomposition* for $G$.
- Treedepth of a graph, denoted $\mathbf{td}(G)$, is the least possible height of a forest into which $G$ can be embedded, where the height of a forest is maximum of heights of its trees.

# Example



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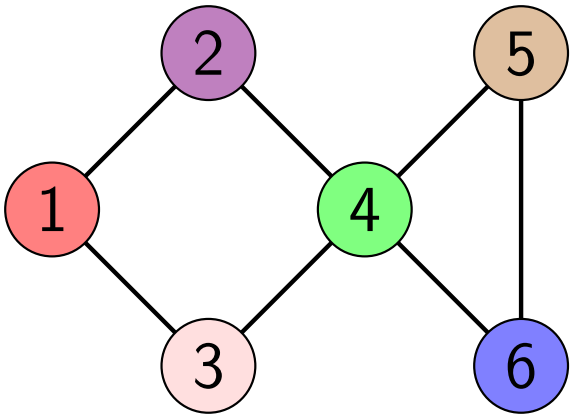


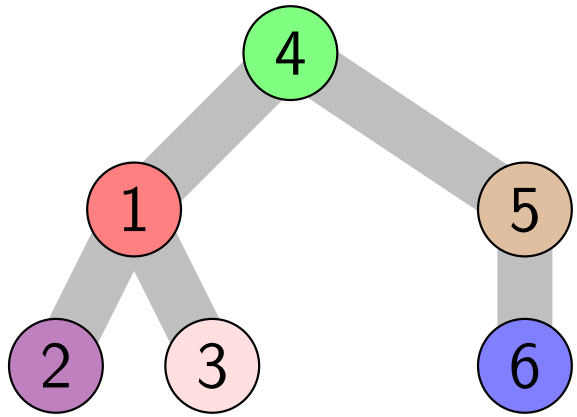

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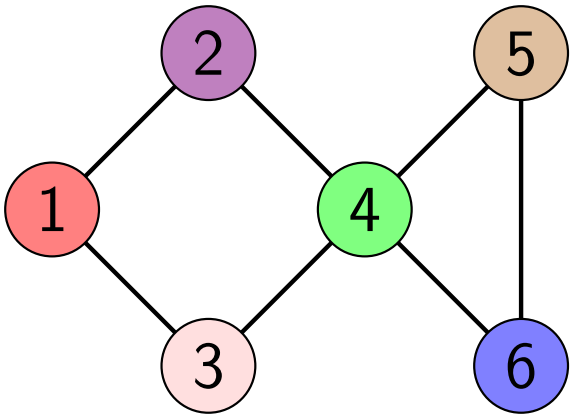


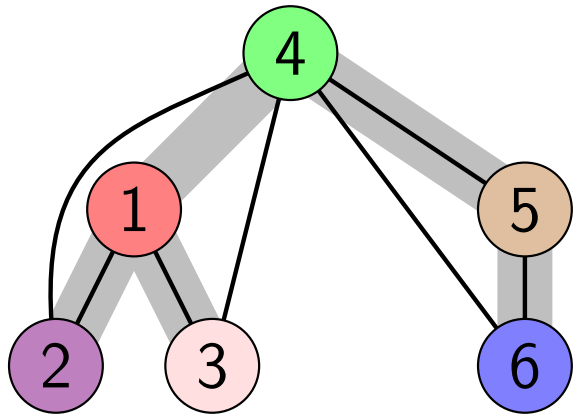

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- $\mathbf{tw}(G) \leqslant \mathbf{td}(G) \leqslant \mathbf{tw}(G) \cdot \log n$
- One of the most overlooked graph parameters that we only now begin to understand.

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- **This talk**: Sketching breaking $2^n$.

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- From now on we concentrate on the case when $G$ is connected, so the decomposition must be a tree.

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- **Note**: a minimal tree has minimum possible height.

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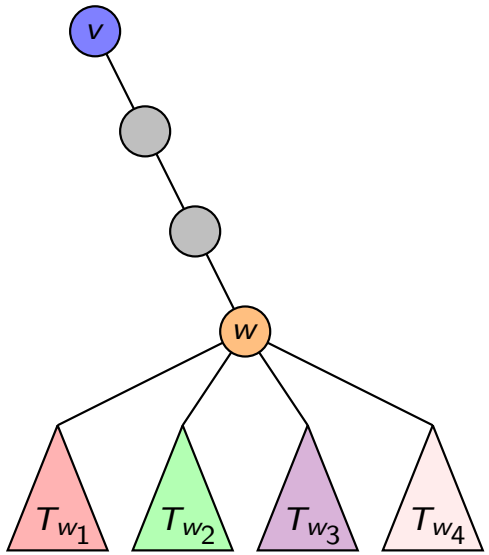
- **Proof sketch**: Otherwise one can partition $T_v$ into two smaller subtrees, and replacing $T_v$ with these subtrees in $T$ gives a decomposition smaller wrt. $\prec$.

# Minimal trees: properties

## Lemma 2

Let $T$ be a minimal tree for $G$ and let $v$ be any vertex. Let $w$ be the nearest branching point of $T$ below $v$ (possibly $v = w$), and let $w_1, w_2, \ldots, w_p$ be children of $w$. Then vertex $v$ has a neighbour in $G$ in each of sets $V(T_{w_1}), V(T_{w_2}), \ldots, V(T_{w_p})$.



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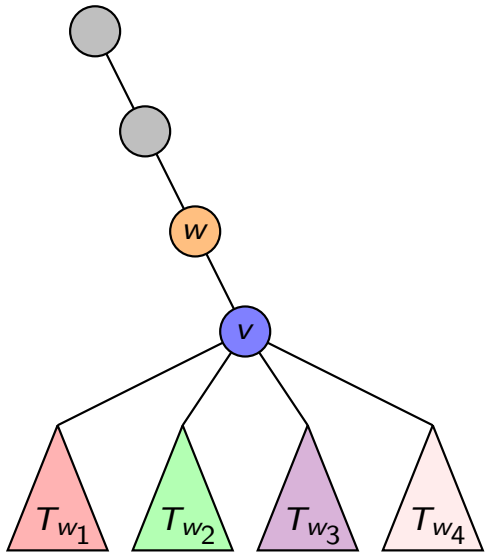

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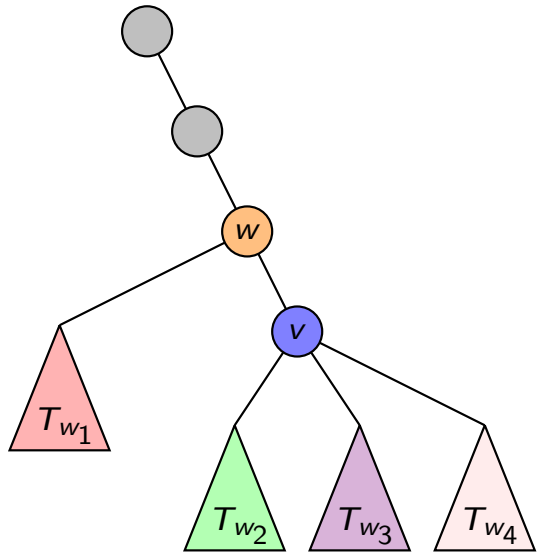

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- **Recurrence**: For $X \subseteq V(G)$ inducing a connected subgraph, the following holds:

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  - Show that if the pruned DP omitted the optimum decomposition, then this decomposition must have a special structure for which there is significantly less than $2^n$ possibilities.

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- Cardinality of $\mathcal{S}$ is at most $\mathcal{O}^\star\left(\binom{n}{(\frac{1}{2}-\varepsilon)\cdot n}\right)$.

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- **Corollary**:
  $\mathbf{td}_*(G[X]) = \mathbf{td}(G[X])$ for all $X$ with $|X| \leqslant (\frac{1}{2} - \varepsilon) \cdot n$.

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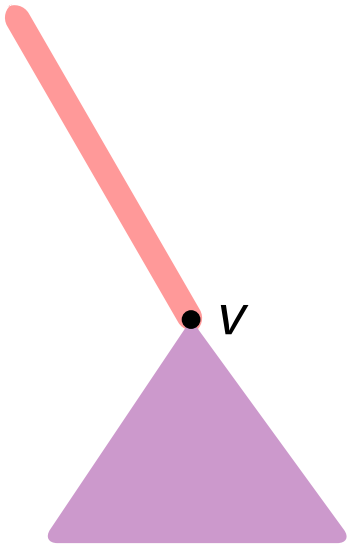

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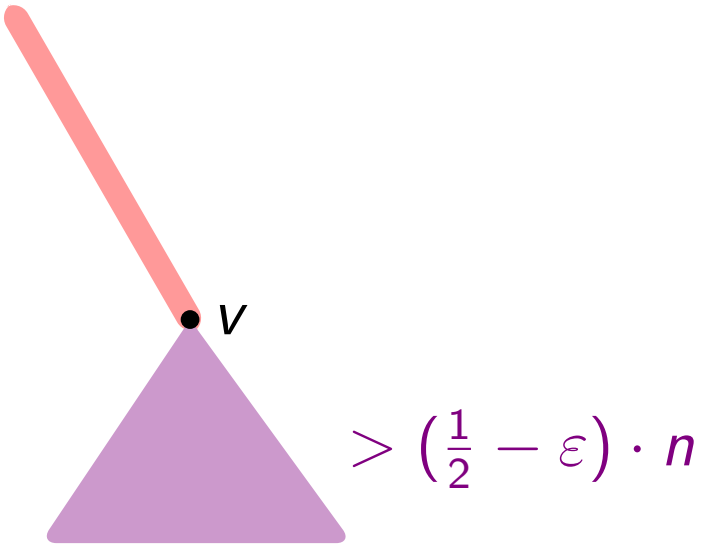

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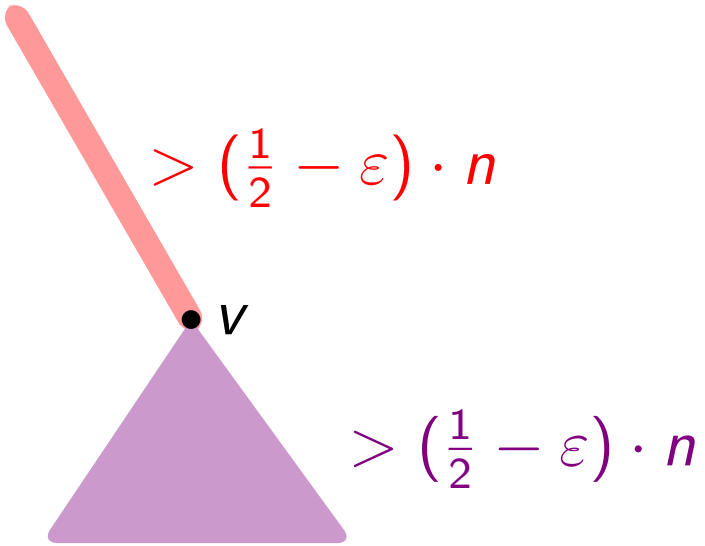

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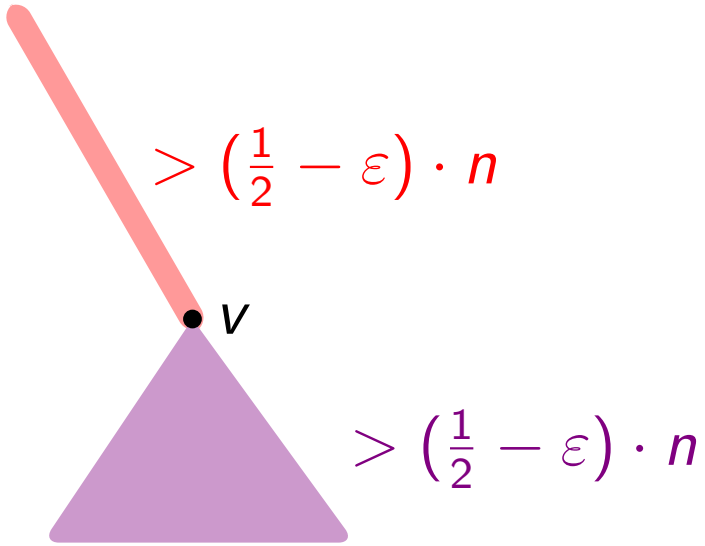

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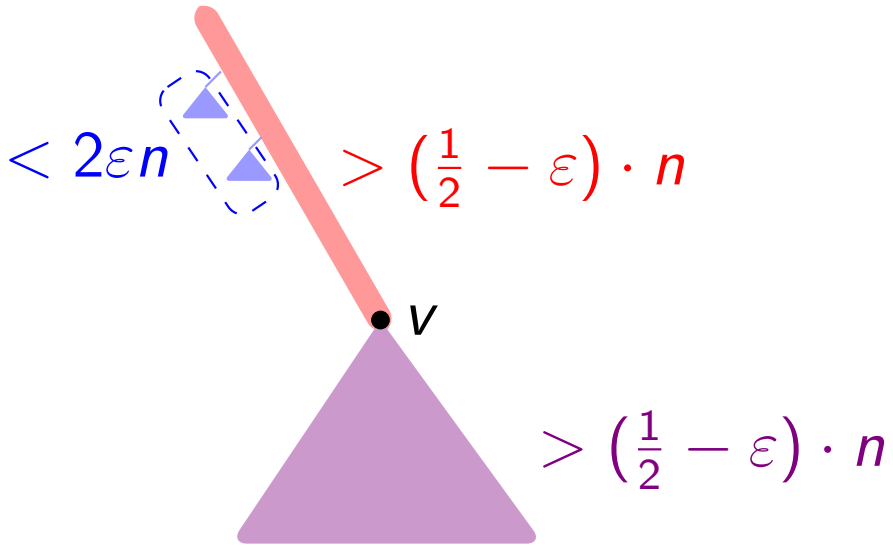

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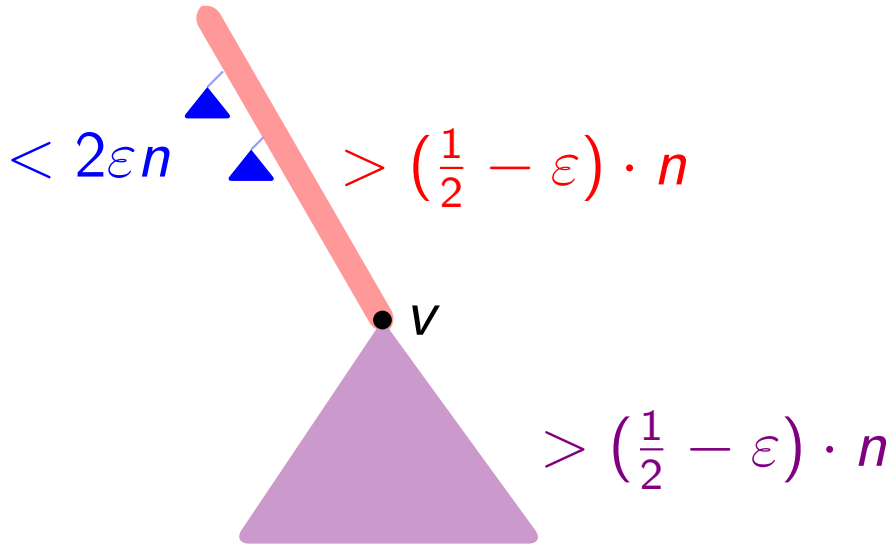

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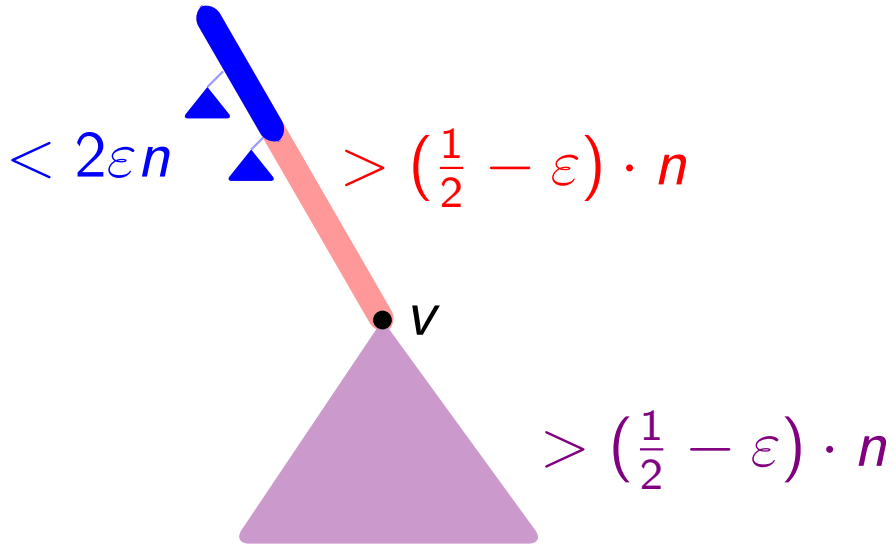

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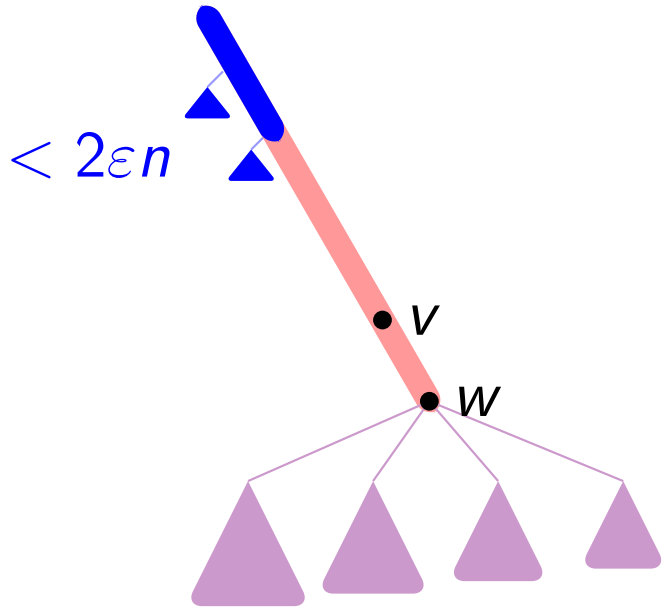

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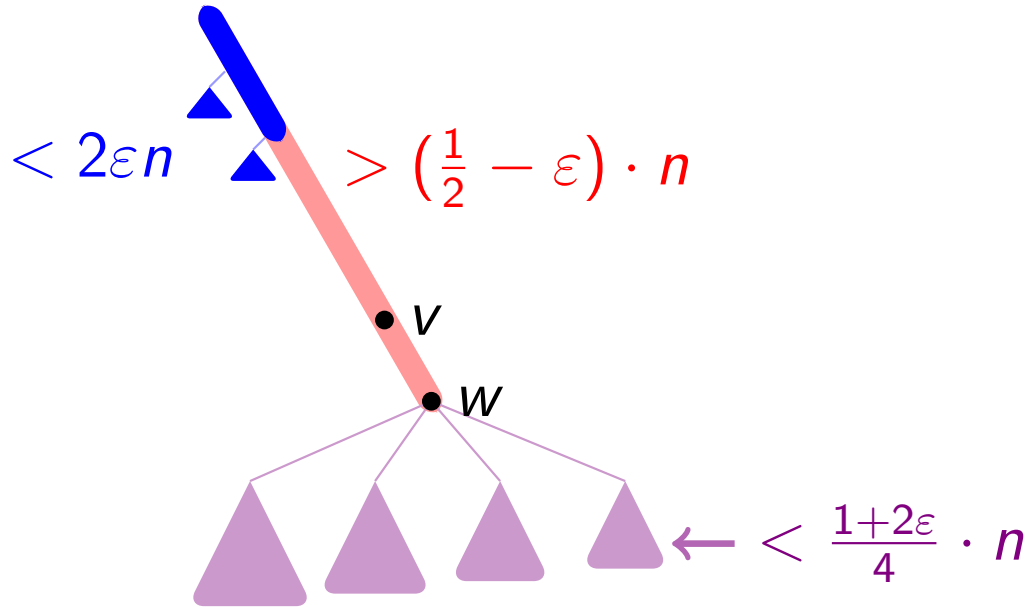

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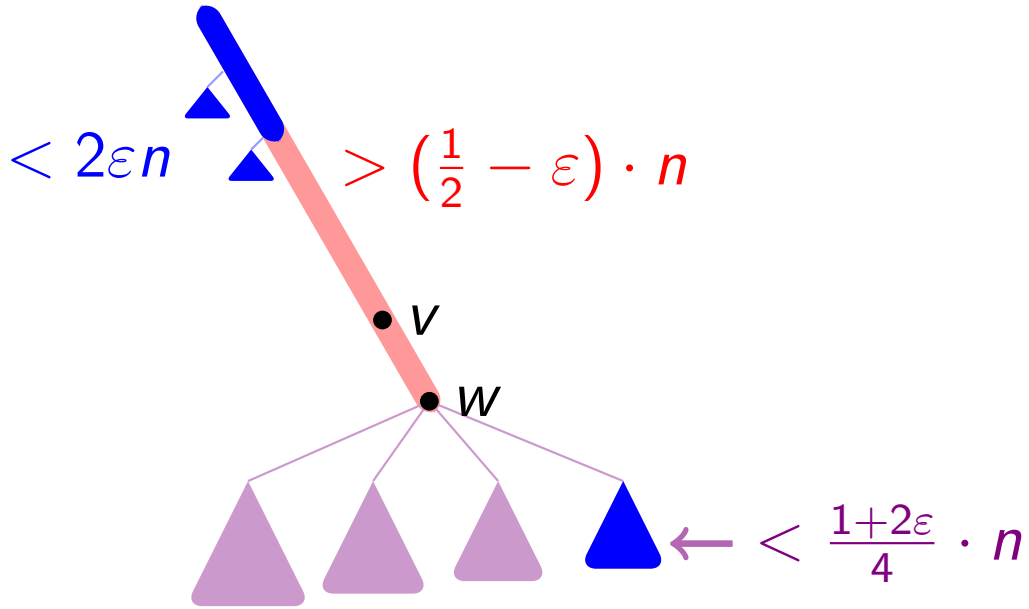

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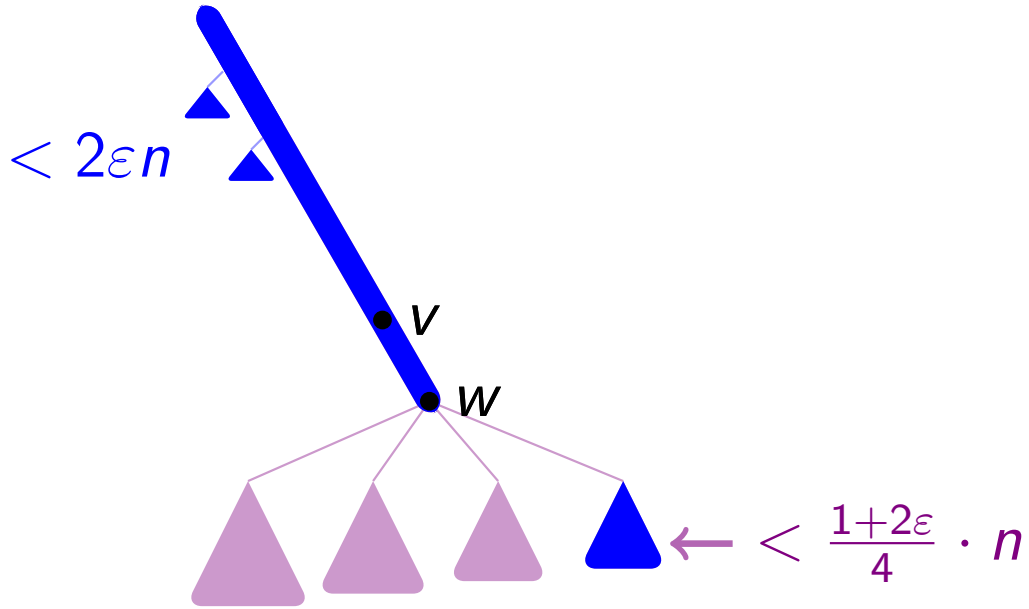

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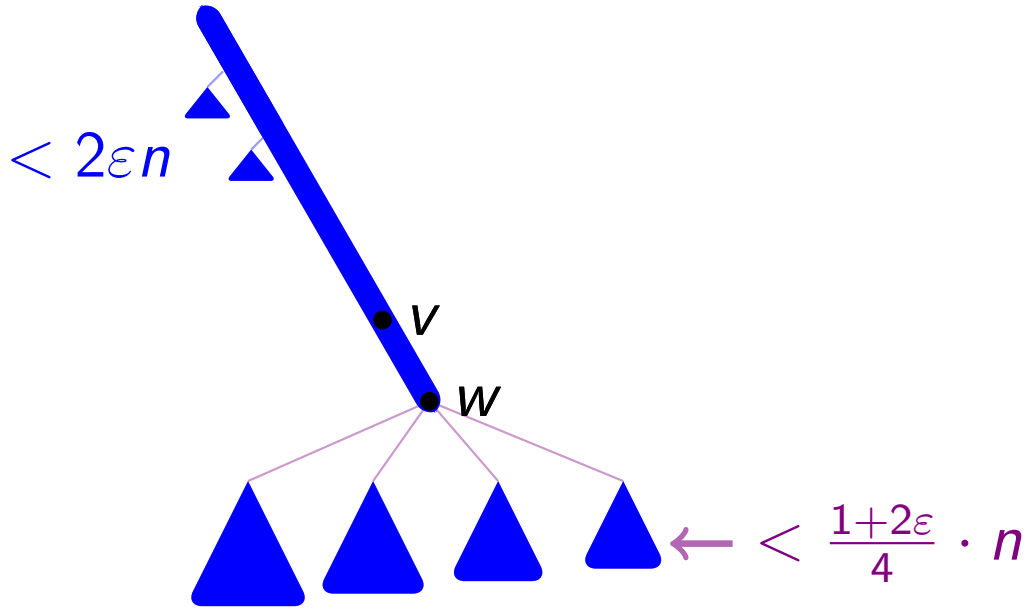

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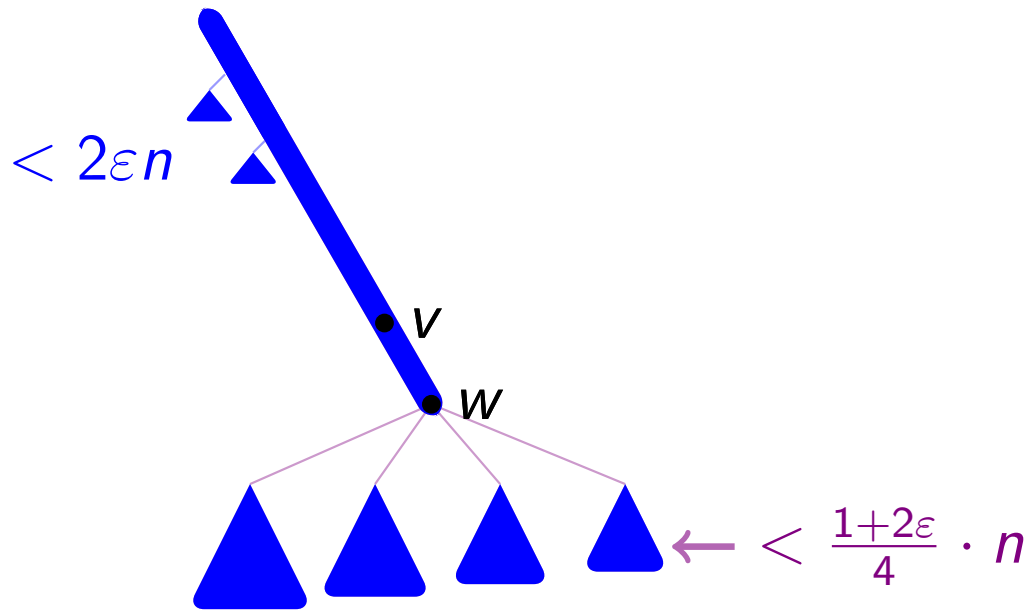


- **Number of choices**: at most $\mathcal{O}^\star\left(\binom{n}{2\varepsilon n} \cdot \binom{n}{\frac{1+2\varepsilon}{4} \cdot n}\right)$.

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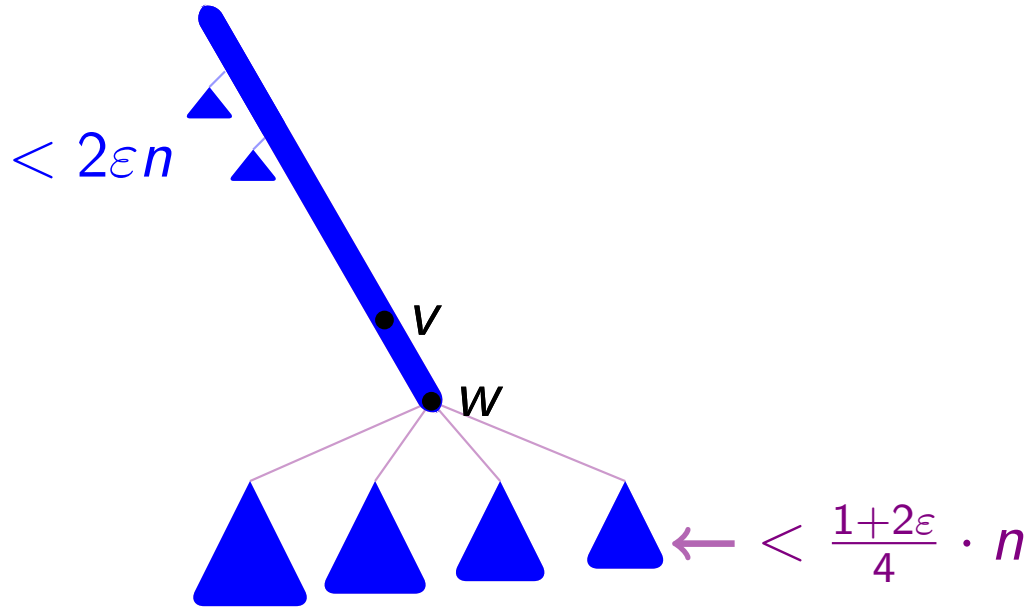


- **Number of choices**: at most $\mathcal{O}^\star\left(\binom{n}{2\varepsilon n} \cdot \binom{n}{\frac{1+2\varepsilon}{4} \cdot n}\right)$.
- For each choice, use $\mathcal{O}^\star\left(\binom{n}{2\varepsilon n}\right)$ time to compute the treedepth of each subtree.

# Conclusions

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- **Thank you!**