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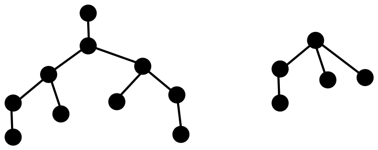

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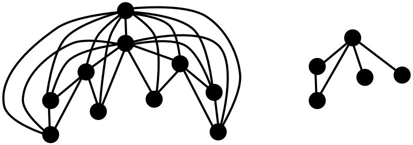

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- Tight connections with graph parameter *treedepth*.
- From now on, all trivially perfect graphs will be connected.

Universal Clique Decomposition

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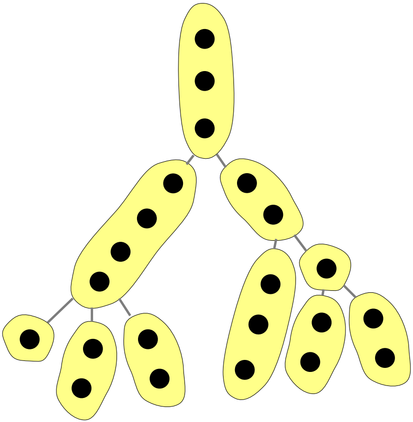
A *universal clique decomposition* of a graph H is a rooted tree $\mathcal{T}$, where every node t is assigned a bag B_t such that:

- $\{B_t \colon t \in V(\mathcal{T})\}$ form a partition of $V(H)$;
- non-leaf nodes of $\mathcal{T}$ have at least two sons;
- $uv \in E(H)$ if and only if u and v share a bag, or their bags are in ancestor-descendant relation.

UC-decomposition on a picture



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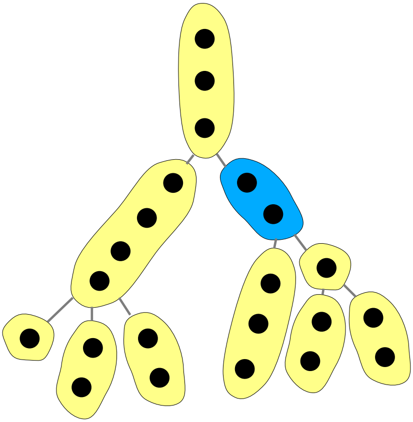

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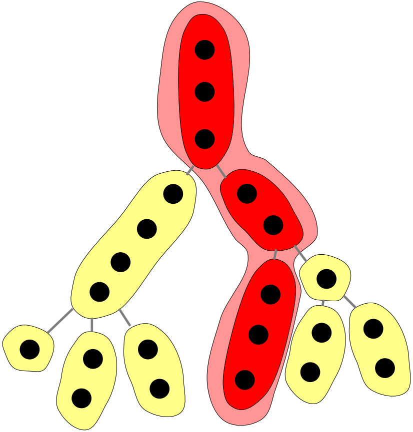

UC-decomposition and TP graphs

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Lemma

A connected graph is trivially perfect if and only if it admits a universal clique decomposition. Moreover, the universal clique decomposition of a trivially perfect graph is unique.

Trivially perfect completion

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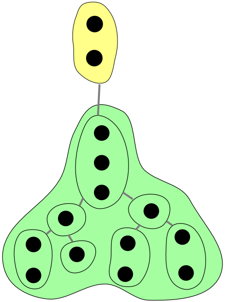
- **WRONG**: Is there a completion set of size $\leq k$ that kills all induced C_4-s and P_4-s?
- **RIGHT**: Can one find a rooted tree of bags that needs at most k fill edges to be turned into a universal clique decomposition?

Minimal completions: properties

Lemma 1

Suppose F is a minimal completion for G, and let C be a subtree of the UC-decomposition of $G + F$. Then $G[V(C)] := G[\bigcup_{t \in V(C)} B_t]$ is connected.



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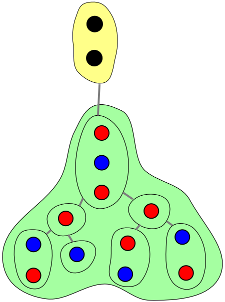

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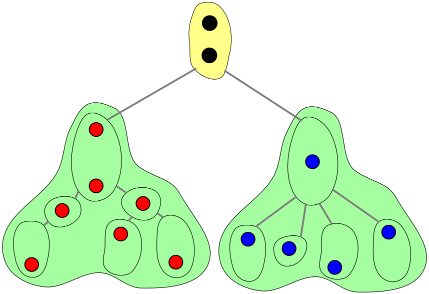
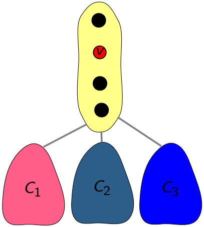


Minimal completions: properties

Lemma 2

Assume F is a minimal completion for G and let v be any vertex. Let $C_1, C_2, \ldots, C_p$ be the subtrees of the UC-decomposition of $G + F$ below the bag of v. Then v has at least one neighbour in each of $V(C_1), V(C_2), \ldots, V(C_p)$ in G.



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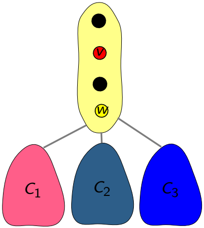

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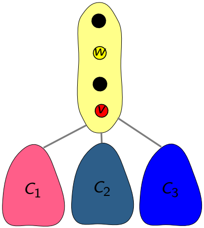

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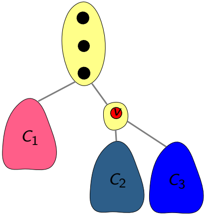
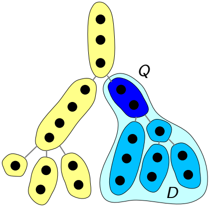


Blocks

Block

For the UC-decomposition $\mathcal{T}$ of a TP graph H and its node t, a *block* at t is the pair (Q, D), where $Q = B_t$ and D is the set of vertices in the subtree of $\mathcal{T}$ rooted in t.



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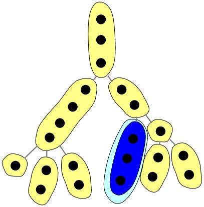

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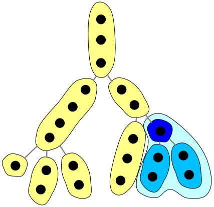

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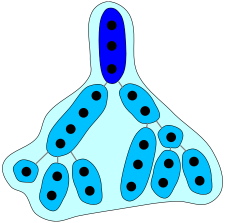

Dynamic programming on potential blocks

- Let $\mathbb{S}_0$ be the set of all pairs (Q, D) such that $Q \subseteq D \subseteq V(G)$ and $G[D]$ is connected. Of course $|\mathbb{S}_0| \leq 3^n$.

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- **Lemma 1** gives the following recurrence:

$$T[Q, D] \;=\; \text{\#edges needed to make } Q \text{ a UC} + \sum_{C \in cc(G[D \setminus Q])} \left(\min_{Q_C \subseteq C} T[Q_C, C] \right)$$

Trimmed dynamic programming

- Exact $\mathcal{O}^\star(4^n)$ algorithm: compute $T[\cdot,\cdot]$ for the whole family $\mathbb{S}_0$, and output $\min_{Q\subseteq V(G)} T[Q, V(G)]$.

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- **Easy**: $T_{\mathbb{S}}[Q, D] \geq T[Q, D]$ for every $(Q, D) \in \mathbb{S}$.
- **Crucial**: Suppose there exists a minimum completion F such that every block of the UC-decomposition of $G + F$ belongs to $\mathbb{S}$. Then

$$\min_{(Q,V(G)) \in \mathbb{S}} T_{\mathbb{S}}[Q, V(G)] = \min_{Q \subseteq V(G)} T[Q, V(G)].$$

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- **Note**: In family $\mathbb{S}$ we just need to hit all the blocks of OPT. We may put into $\mathbb{S}$ a lot of unnecessary states as well, and they do not make any harm.

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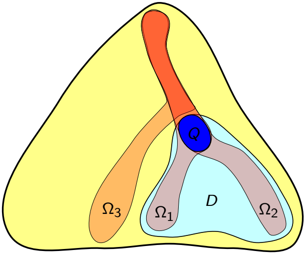
- Let (Q, D) be a block and $\Omega_1, \Omega_2, \Omega_3$ be as on the figure.



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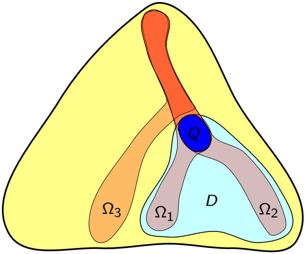
- Let (Q, D) be a block and $\Omega_1, \Omega_2, \Omega_3$ be as on the figure.
- Then $Q = (\Omega_1 \cap \Omega_2) \setminus \Omega_3$,



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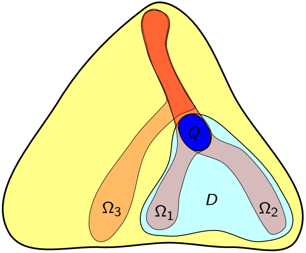
- and D is the vertex set of the connected component of $G \setminus \Omega_3$ that contains Q.



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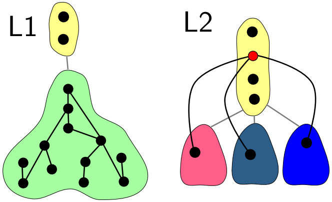
  - We are going to find such a family of size $k^{\mathcal{O}(\sqrt{k})}$. Let us fix some optimum completion F with $|F| \leq k$.

Turn to blackboard



Polynomial kernel: There exists a cubic kernel of Guo for TP-Completion. Hence we can assume that $n = \mathcal{O}(k^3)$.

Goal: A family $\mathbb{P}$ of $k^{\mathcal{O}(\sqrt{k})}$ subsets of $V(G)$ such that every maximal clique in $G + F$ belongs to $\mathbb{P}$.