Jungles, bundles, and fixed parameter tractability

Fedor Fomin and Michał Pilipczuk

Department of Informatics
University of Bergen

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The background

MINOR/TOPOLOGICAL SUBGRAPH CONTAINMENT

Input: Undirected graphs H and G.
Question: Is H contained in G as a minor/topological subgraph?

The algorithms

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- For MINOR CONTAINMENT,
  $f(|H|)|V(G)|^3$ algorithm [Robertson and Seymour, 1995].

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  Polynomial time algorithm for a fixed H with constant exponent, i.e., $f(|H|)|G|^c$ for some (small) constant c.
- For Minor Containment,
  $f(|H|)|V(G)|^3$ algorithm [Robertson and Seymour, 1995].
- For Topological Subgraph Containment,
  $f(|H|)|V(G)|^3$ algorithm [Grohe et al., 2011].

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- In acyclic digraphs there is an XP algorithm, but FPT is unlikely [Slivkins, 2010].
- There is hope, when G is a tournament.

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- **Our results:** Refining recent work of Fradkin and Seymour to get FPT algorithm for topological containment ($O(|V|^5)$).

Approach of Fradkin and Seymour

$$\Big\downarrow H, T$$

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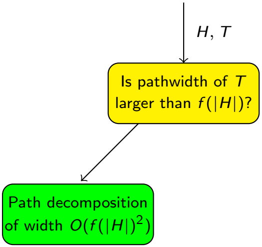
$$\downarrow H,\ T$$

Is pathwidth of T larger than $f(|H|)$?

Approach of Fradkin and Seymour



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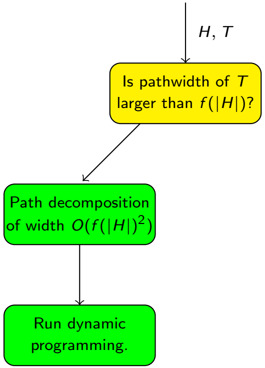

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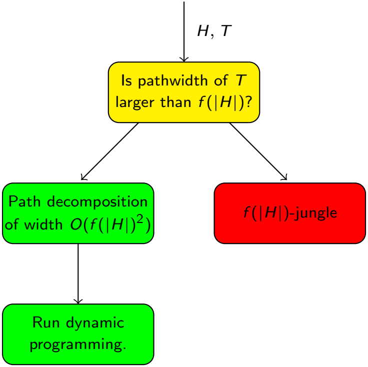

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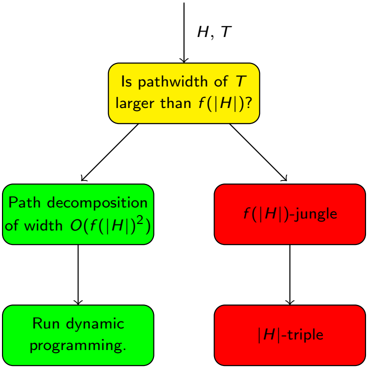

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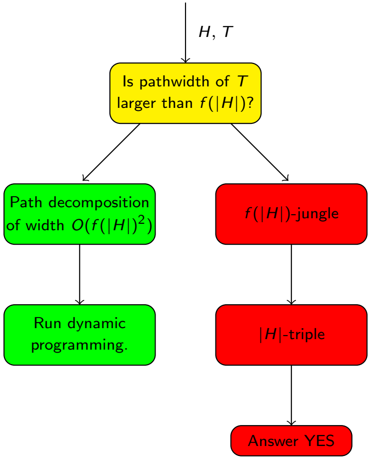

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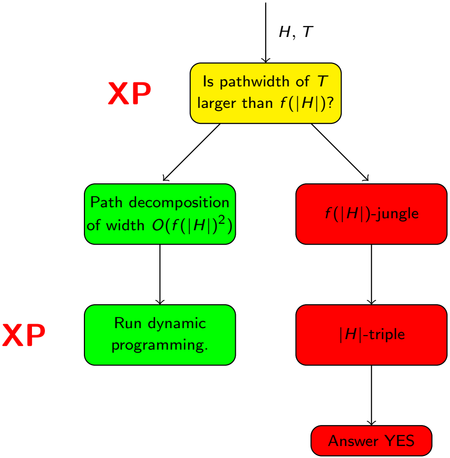

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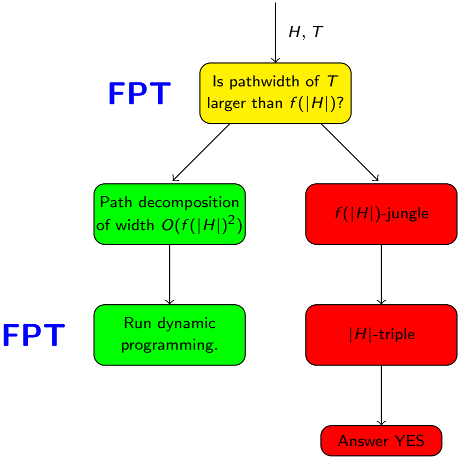

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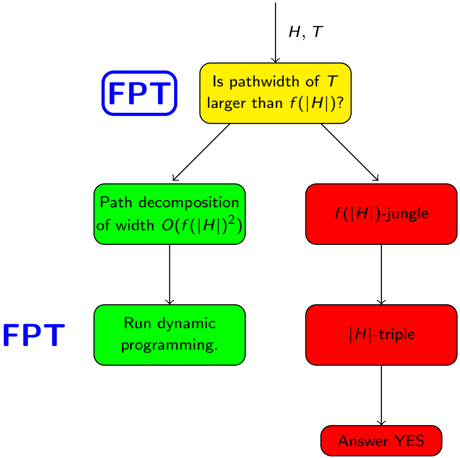


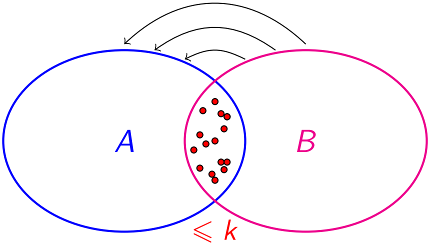
The result

FPT approximation of tournament pathwidth

There exists an algorithm, which given a tournament T and an integer k, outputs either a path decomposition of T of width at most $4k^2 + 7k$, or a k-jungle in T, in time complexity $2^{O(k^2)}|V(T)|^5$.

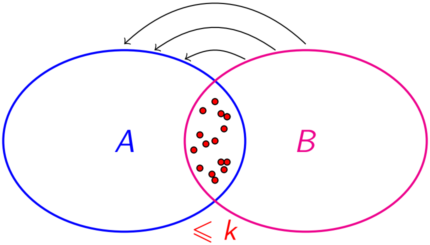
Separation

- (A, B) is **a separation of order k if**



Separation

- (A, B) is **a separation of order** k if
  - $A \cup B = V(T)$, $|A \cap B| \leqslant k$;



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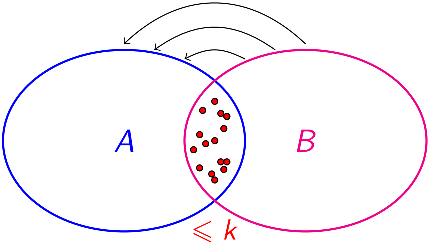

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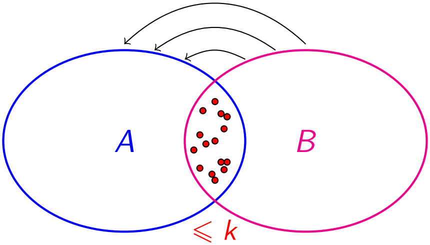
  - $A \cup B = V(T)$, $|A \cap B| \leqslant k$;
  - and there are no edges from $A \setminus B$ to $B \setminus A$.
- Separations (A, B) and (C, D) **do not cross** if $A \subseteq C$ and $D \subseteq B$ or vice versa.



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- **pw**$(T) = \max W_i - 1$.
- k-**jungle**: set X of cardinality k, such that every two vertices of X are k-connected.

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- We greedily incorporate bigger and bigger separations of the tournament up to order k, constructing a cross-free family of separations called a **bundle**.

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  - **Small width**: we are happy.
  - **Large width**: a k-jungle due to maximality of the bundle.

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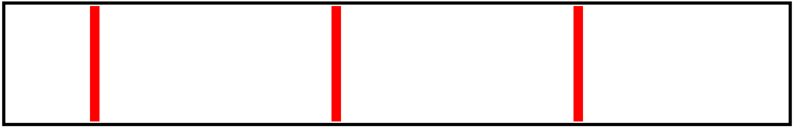

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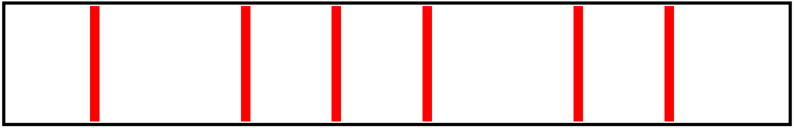

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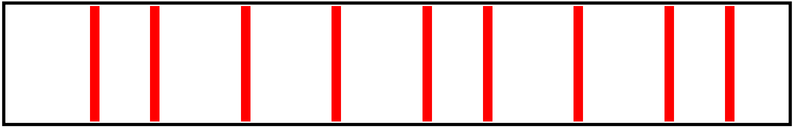

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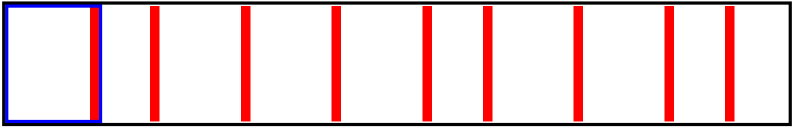

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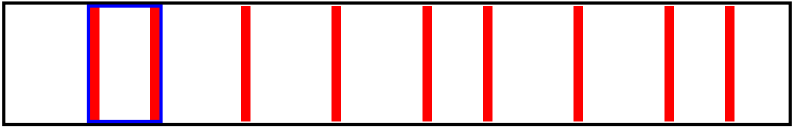

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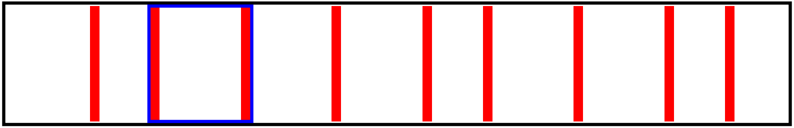

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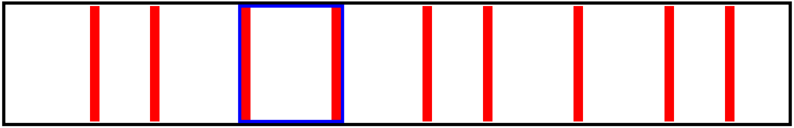

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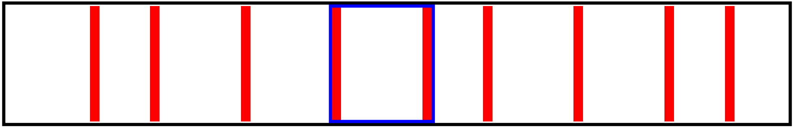

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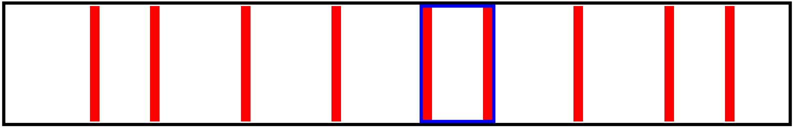

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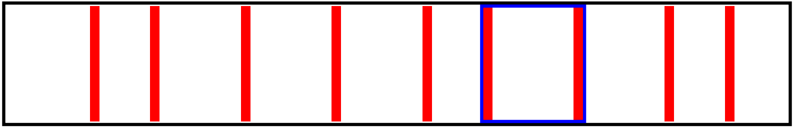

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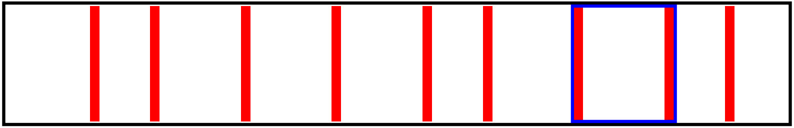

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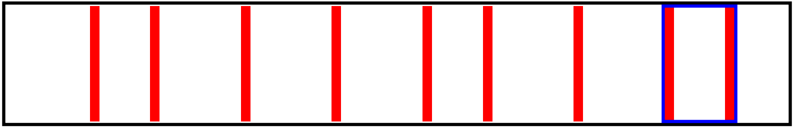

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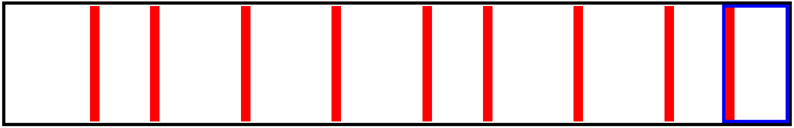
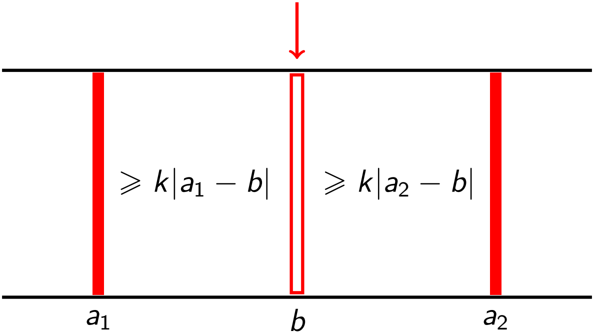


Inserting new separation

- The new separation cannot be 'close' to the neighbouring ones.



Inserting new separation

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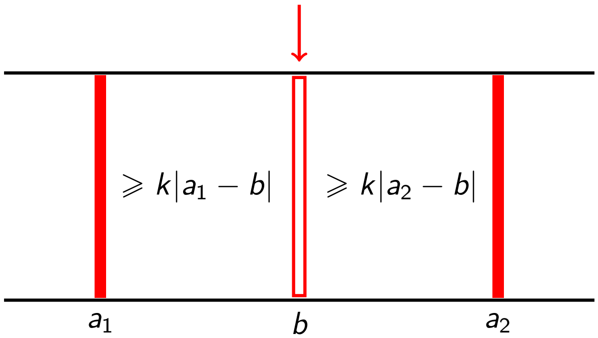
- There have to be at least $k|a_1 - b|$, $k|a_2 - b|$ vertices in between, respectively.



Subproblem

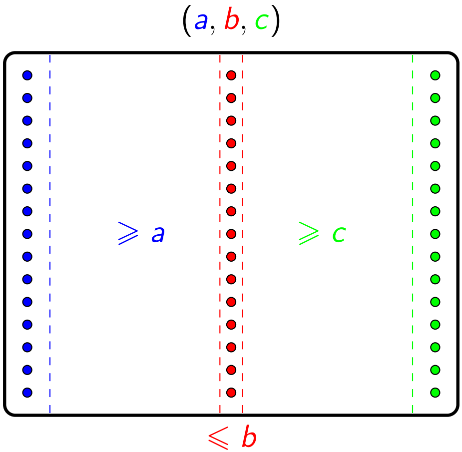
Tournament Balanced Separator

Input: A tournament T; disjoint sets $X, Y \subseteq V(T)$; integers a, b, c.
Question: Does there exist a separation (A, B) of T such that

- $|A \cap B| \leqslant k$;
- $X \subseteq A \setminus B$, $Y \subseteq B \setminus A$;
- $|A \setminus (X \cup B)| \geqslant a$ and $|B \setminus (Y \cup A)| \geqslant c$?

We show an algorithm working in time $2^{O(a+b+c)}|V(T)|^4$.

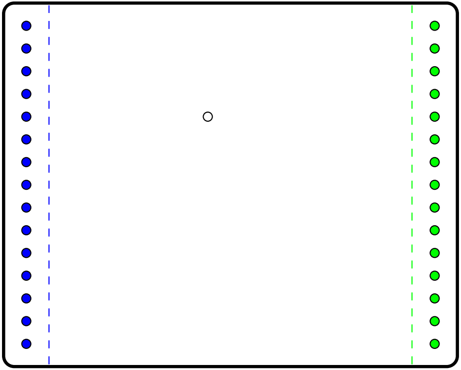
Subproblem



Branching

Take any vertex and branch on it:

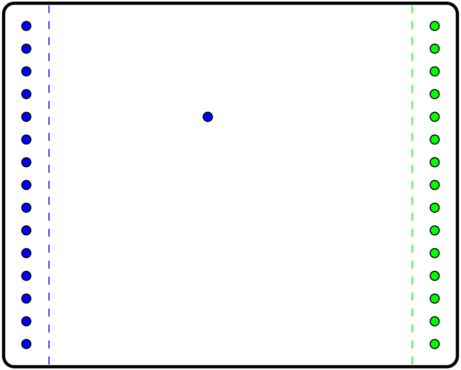
(a, b, c)



Branching

Take any vertex and branch on it:
Goes to $A \setminus B$.

(a, b, c)



Branching

Take any vertex and branch on it:
Goes to $B \setminus A$.

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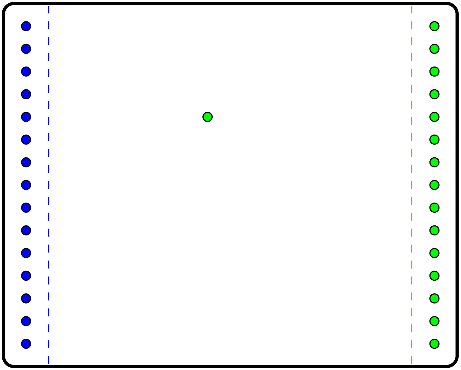

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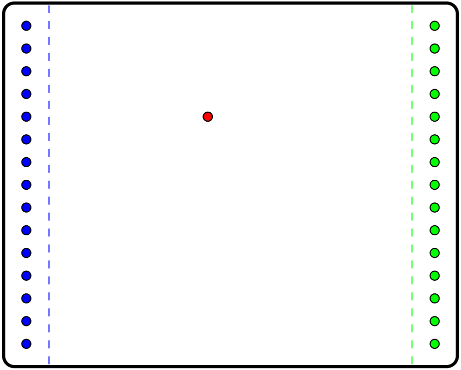
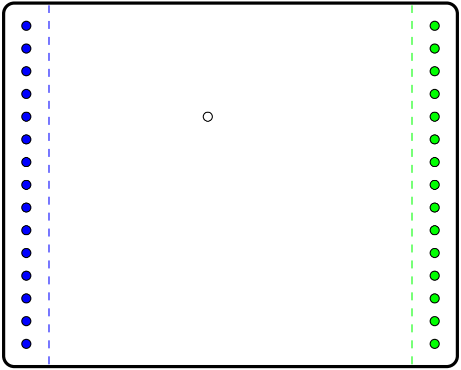


Branching

Take any vertex and branch on it:
If we run out of vertices: OK!

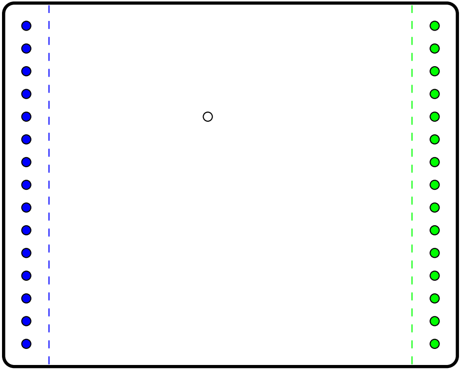
(a, b, c)



Branching

Take any vertex and branch on it:
If we run out of b: OK!

(a, b, c)

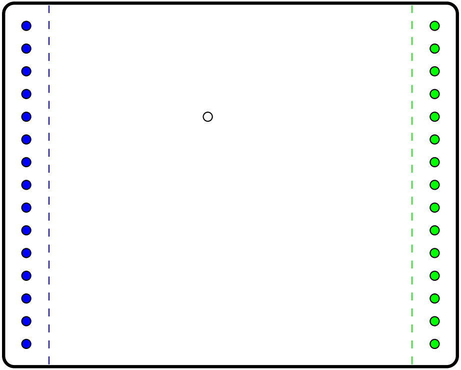


Branching

Take any vertex and branch on it:

By symmetry, we run out of a.

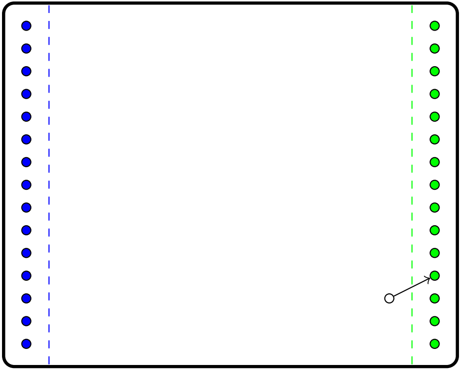
(a, b, c)



Branching

Take any vertex with outneighbour in Y and branch on it:

$(0, b, c)$



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Goes to $B \setminus A$.

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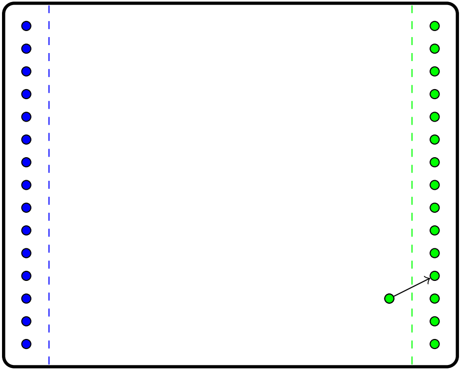

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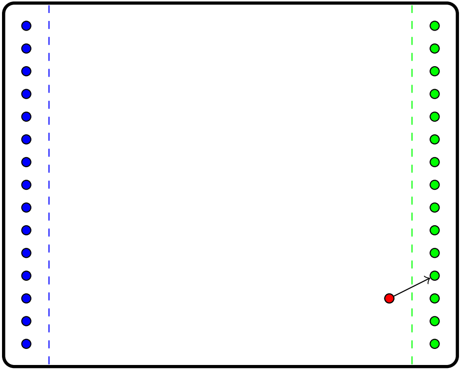
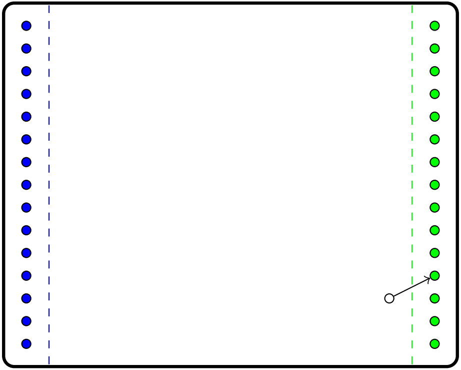


Branching

Take any vertex with outneighbour in Y and branch on it:

If we run out of b: OK!

$(0, b, c)$

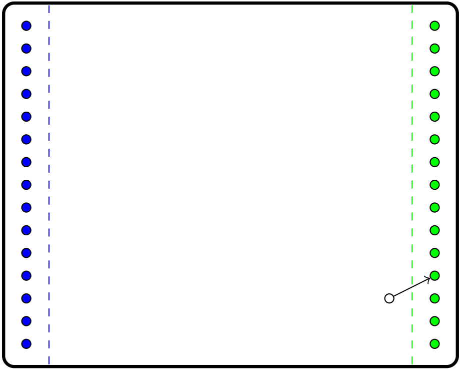


Branching

Take any vertex with outneighbour in Y and branch on it:

If we run out of c: OK!

$(0, b, c)$

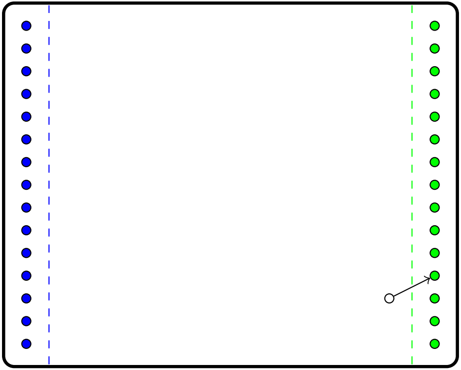


Branching

Take any vertex with outneighbour in Y and branch on it:

Then we run out of vertices...

$(0, b, c)$



Reduced problem

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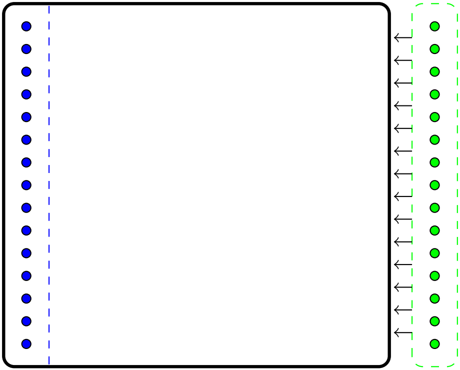

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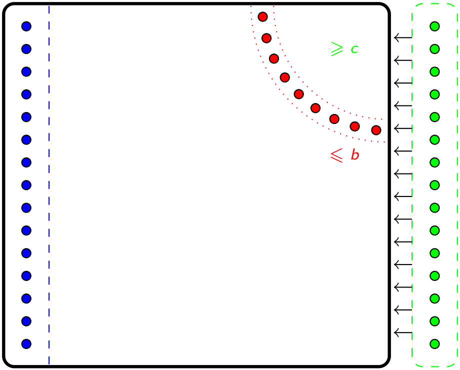

Subsubproblem

Tournament Subset Separation

Input: A tournament T with set of terminals X; integers b, c.
Question: Does there exist a separation (A, B) of T of order b, such that $X \subseteq A \setminus B$ and $|B \setminus A| \geqslant c$?

Subsubproblem: solution

- Now comes the tricky part.

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  - $|B \setminus A| \geqslant 2b + 2c$;

Subsubproblem: solution

- Now comes the tricky part.
- We consider two cases:
  - $|B \setminus A| \geqslant 2b + 2c$;
  - $|B \setminus A| \leqslant 2b + 2c - 1$.

$|B \setminus A| \geqslant 2b + 2c$

- Consider subtournament $T[B \setminus A]$; there must be a vertex v with indegree at least $b + c - 1$.

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- Branch into $|V(T) \setminus X|$ subcases, in each taking different nonterminal as v.
- We compute the minimum cut between X and v:

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- The separation we have found can be different than (A, B), but it suffices to our needs.

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- We do brute-force: iterate through all the subsets of these vertices of small indegree.

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- This suggests change of the width parameter.

Thank you

Questions?