Jungles, bundles, and fixed parameter tractability

Fedor Fomin and Michał Pilipczuk

Department of Informatics
University of Bergen

SODA 2013, January 6^{th}

The basic problem

Minor/Topological Subgraph Containment

Input: Undirected graphs H and G.
Question: Is H contained in G as a minor/topological subgraph?

The algorithms

- **First goal [XP]:**
  Polynomial time algorithm for every fixed H, e.g., $O(|G|^{|H|})$.

The algorithms

- **First goal [XP]:**
  Polynomial time algorithm for every fixed H, e.g., $O(|G|^{|H|})$.
- **Second goal [FPT]:**
  Polynomial time algorithm for every fixed H, with exponent independent of H, i.e., $f(|H|)|G|^c$ for some (small) constant c.

The algorithms

- **First goal [XP]:**
  Polynomial time algorithm for every fixed H, e.g., $O(|G|^{|H|})$.
- **Second goal [FPT]:**
  Polynomial time algorithm for every fixed H, with exponent independent of H, i.e., $f(|H|)|G|^c$ for some (small) constant c.
- For Minor Containment,
  $f(|H|)|V(G)|^3$ algorithm [Robertson and Seymour].

The algorithms

- **First goal [XP]:**
  Polynomial time algorithm for every fixed H, e.g., $O(|G|^{|H|})$.
- **Second goal [FPT]:**
  Polynomial time algorithm for every fixed H, with exponent independent of H, i.e., $f(|H|)|G|^c$ for some (small) constant c.
- For Minor Containment,
  $f(|H|)|V(G)|^3$ algorithm [Robertson and Seymour].
- For Topological Subgraph Containment,
  $f(|H|)|V(G)|^3$ algorithm [Grohe et al.].

Directed world

- We consider topological subgraph and immersion relations.

Directed world

- We consider topological subgraph and immersion relations.
  - TOPOLOGICAL SUBGRAPH: vertices of H map to different vertices in G, and arcs in H map to **vertex-disjoint** directed paths between corresponding images in G.

Directed world

- We consider topological subgraph and immersion relations.
  - TOPOLOGICAL SUBGRAPH: vertices of H map to different vertices in G, and arcs in H map to **vertex-disjoint** directed paths between corresponding images in G.
  - IMMERSION: vertices of H map to different vertices in G, and arcs in H map to **edge-disjoint** directed paths between corresponding images in G.

Directed world

- We consider topological subgraph and immersion relations.
  - Topological subgraph: vertices of H map to different vertices in G, and arcs in H map to **vertex-disjoint** directed paths between corresponding images in G.
  - Immersion: vertices of H map to different vertices in G, and arcs in H map to **edge-disjoint** directed paths between corresponding images in G.
- NP-hard in general setting even for small, fixed subgraphs H [Fortune et al.].

Directed world

- We consider topological subgraph and immersion relations.
  - TOPOLOGICAL SUBGRAPH: vertices of H map to different vertices in G, and arcs in H map to **vertex-disjoint** directed paths between corresponding images in G.
  - IMMERSION: vertices of H map to different vertices in G, and arcs in H map to **edge-disjoint** directed paths between corresponding images in G.
- NP-hard in general setting even for small, fixed subgraphs H [Fortune et al.].
- For acyclic digraphs there is an XP algorithm, but FPT is unlikely [Slivkins].

Tournament world

- Tournaments identified as a class of digraphs where a sound containment theory can be constructed [Chudnovsky, Fradkin, Kim, Scott, and Seymour].

Tournament world

- Tournaments identified as a class of digraphs where a sound containment theory can be constructed [Chudnovsky, Fradkin, Kim, Scott, and Seymour].
- In fact all the results hold for more general *semi-complete digraphs*, but for simplicity assume tournaments in this talk.

Tournament world

- Tournaments identified as a class of digraphs where a sound containment theory can be constructed [Chudnovsky, Fradkin, Kim, Scott, and Seymour].
- In fact all the results hold for more general *semi-complete digraphs*, but for simplicity assume tournaments in this talk.
- A number of FPT algorithms (immersion) and XP algorithms (topological containment).

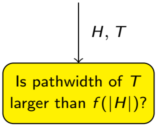
Tournament world

- Tournaments identified as a class of digraphs where a sound containment theory can be constructed [Chudnovsky, Fradkin, Kim, Scott, and Seymour].
- In fact all the results hold for more general *semi-complete digraphs*, but for simplicity assume tournaments in this talk.
- A number of FPT algorithms (immersion) and XP algorithms (topological containment).
- **Our goal:** Refine the running time of algorithms around the topological subgraph problem from XP to FPT.

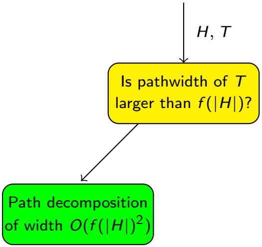
XP algorithm of Fradkin and Seymour

$$\Big\downarrow{\scriptstyle H,\,T}$$

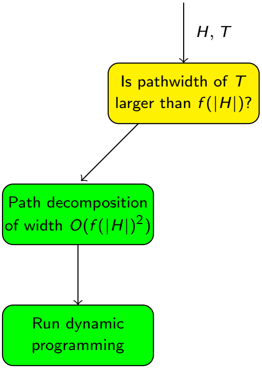
XP algorithm of Fradkin and Seymour



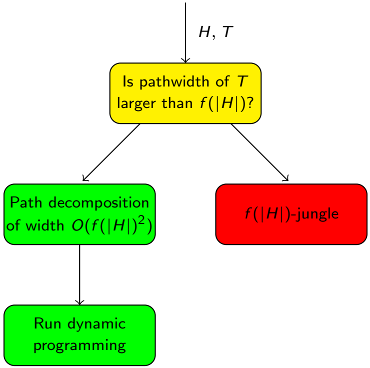
XP algorithm of Fradkin and Seymour



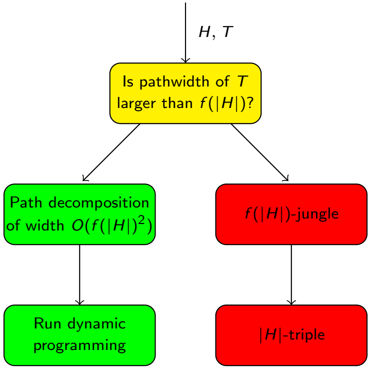
XP algorithm of Fradkin and Seymour



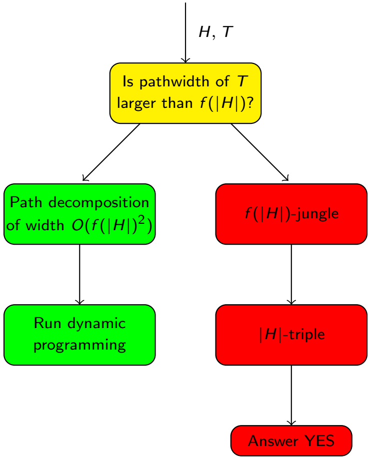
XP algorithm of Fradkin and Seymour



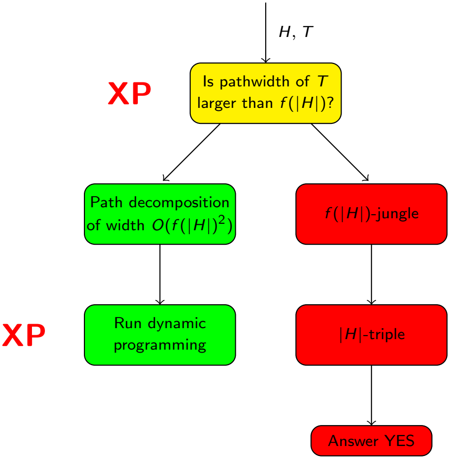
XP algorithm of Fradkin and Seymour



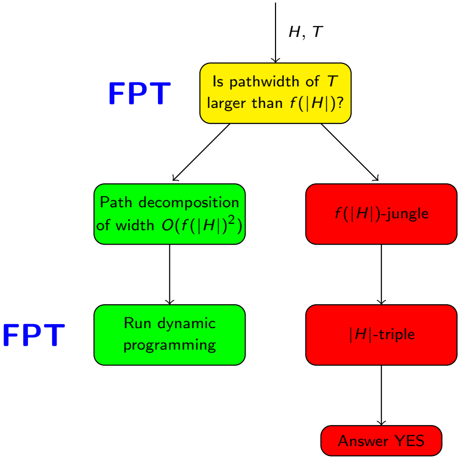
XP algorithm of Fradkin and Seymour



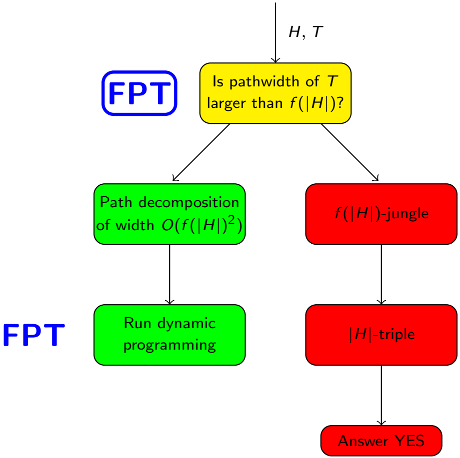
XP algorithm of Fradkin and Seymour



XP algorithm of Fradkin and Seymour



XP algorithm of Fradkin and Seymour



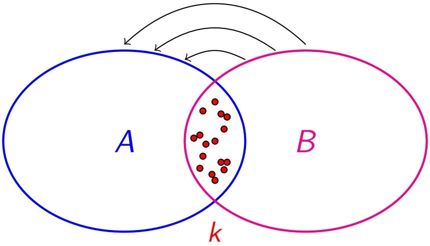
The main result

FPT approximation of pathwidth of a tournament

There exists an algorithm, which given a tournament T on n vertices and an integer k, outputs either a path decomposition of T of width at most $4k^2 + 7k$, or a k-jungle in T, in time complexity $2^{O(k \log k)} \cdot n^3 \log n$.

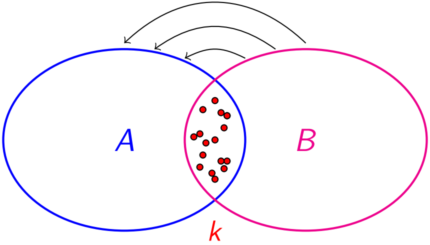
Separation

- (A, B) is **a separation of order** k if



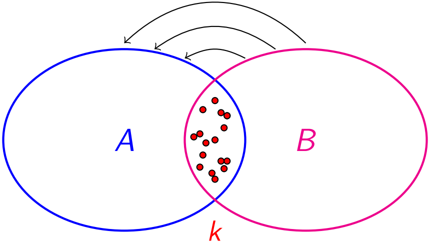
Separation

- (A, B) is **a separation of order** k if
  - $A \cup B = V(T)$, $|A \cap B| = k$;



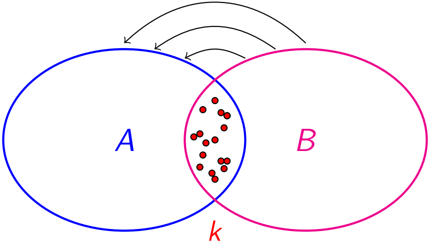
Separation

- (A, B) is **a separation of order** k if
  - $A \cup B = V(T)$, $|A \cap B| = k$;
  - and there are no edges from $A \setminus B$ to $B \setminus A$.



Separation

- (A, B) is **a separation of order** k if
  - $A \cup B = V(T)$, $|A \cap B| = k$;
  - and there are no edges from $A \setminus B$ to $B \setminus A$.
- Separations (A, B) and (C, D) **do not cross** if $A \subseteq C$ and $D \subseteq B$ or vice versa.



Pathwidth

- **Path decomposition** of T is a sequence of bags $[W_1, W_2, \ldots, W_h]$ such that

Pathwidth

- **Path decomposition** of T is a sequence of bags $[W_1, W_2, \ldots, W_h]$ such that
  - $\bigcup W_i = V(T)$;

Pathwidth

- **Path decomposition** of T is a sequence of bags $[W_1, W_2, \ldots, W_h]$ such that
  - $\bigcup W_i = V(T)$;
  - $W_i \cap W_k \subseteq W_j$ for $i < j < k$;

Pathwidth

- **Path decomposition** of T is a sequence of bags $[W_1, W_2, \ldots, W_h]$ such that
  - $\bigcup W_i = V(T)$;
  - $W_i \cap W_k \subseteq W_j$ for $i < j < k$;
  - for every edge (u, v), either $u, v \in W_i$ for some i, or $u \in W_i, v \in W_j$ for some $i > j$.

Pathwidth

- **Path decomposition** of T is a sequence of bags $[W_1, W_2, \ldots, W_h]$ such that
  - $\bigcup W_i = V(T)$;
  - $W_i \cap W_k \subseteq W_j$ for $i < j < k$;
  - for every edge (u, v), either $u, v \in W_i$ for some i, or $u \in W_i, v \in W_j$ for some $i > j$.
- *Width* of $[W_1, W_2, \ldots, W_h]$ is $\max |W_i| - 1$.

Pathwidth

- **Path decomposition** of T is a sequence of bags $[W_1, W_2, \ldots, W_h]$ such that
  - $\bigcup W_i = V(T)$;
  - $W_i \cap W_k \subseteq W_j$ for $i < j < k$;
  - for every edge (u, v), either $u, v \in W_i$ for some i, or $u \in W_i, v \in W_j$ for some $i > j$.
- *Width* of $[W_1, W_2, \ldots, W_h]$ is $\max |W_i| - 1$.
- **pw**(T) is the minimum possible width of a decomposition.

Approximation algorithm of Fradkin and Seymour

- We greedily incorporate separations of larger and larger order up to order k, constructing a cross-free family of separations called a **bundle**.

Approximation algorithm of Fradkin and Seymour

- We greedily incorporate separations of larger and larger order up to order k, constructing a cross-free family of separations called a **bundle**.
- Each new separation has to satisfy certain technical conditions.

Approximation algorithm of Fradkin and Seymour

- We greedily incorporate separations of larger and larger order up to order k, constructing a cross-free family of separations called a **bundle**.
- Each new separation has to satisfy certain technical conditions.
- Having a maximum bundle we obtain some path decomposition:

Approximation algorithm of Fradkin and Seymour

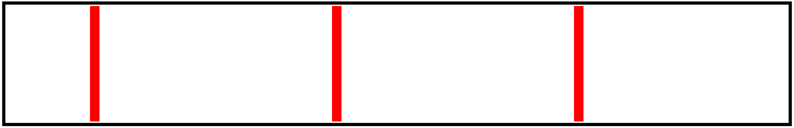
- We greedily incorporate separations of larger and larger order up to order k, constructing a cross-free family of separations called a **bundle**.
- Each new separation has to satisfy certain technical conditions.
- Having a maximum bundle we obtain some path decomposition:
  - **Small width**: we are happy.

Approximation algorithm of Fradkin and Seymour

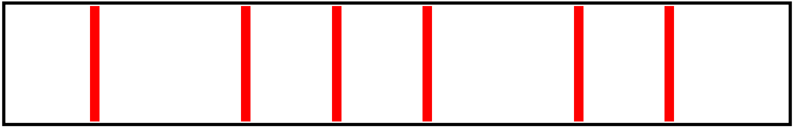
- We greedily incorporate separations of larger and larger order up to order k, constructing a cross-free family of separations called a **bundle**.
- Each new separation has to satisfy certain technical conditions.
- Having a maximum bundle we obtain some path decomposition:
  - **Small width**: we are happy.
  - **Large width**: a k-jungle due to maximality of the bundle.

Algorithm: overview

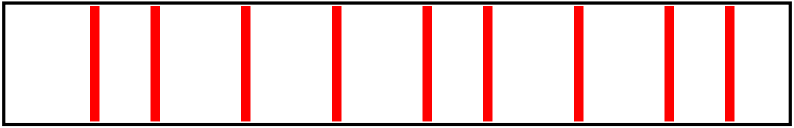
Algorithm: overview



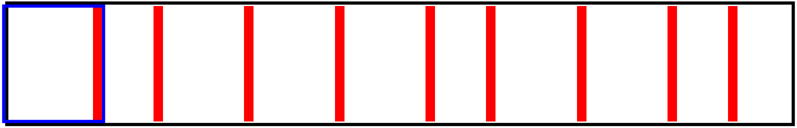
Algorithm: overview



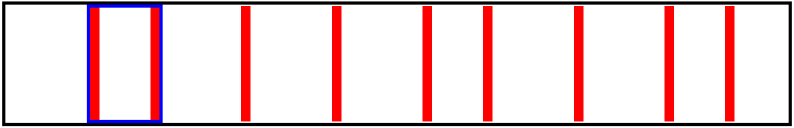
Algorithm: overview



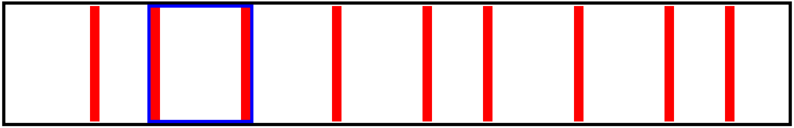
Algorithm: overview



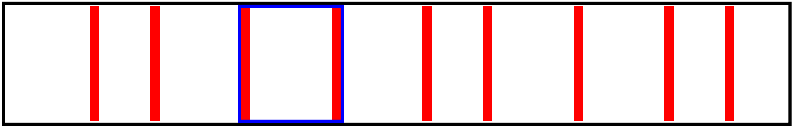
Algorithm: overview



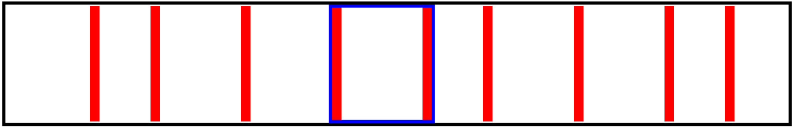
Algorithm: overview



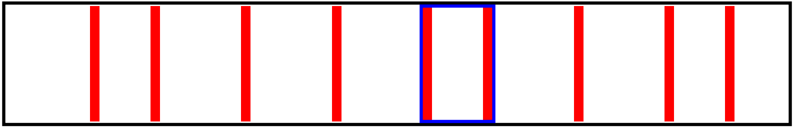
Algorithm: overview



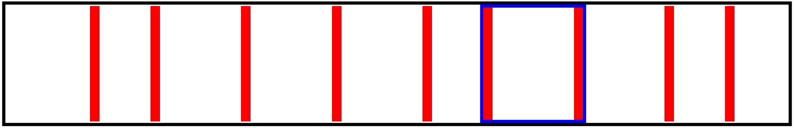
Algorithm: overview



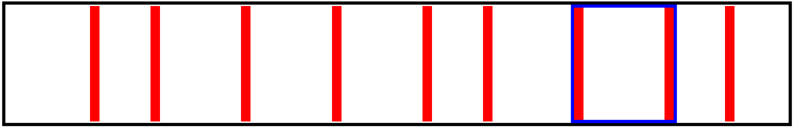
Algorithm: overview



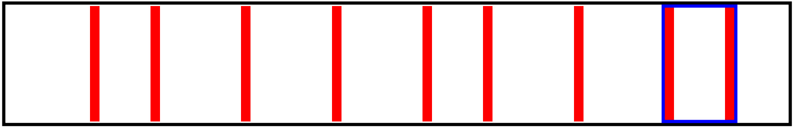
Algorithm: overview



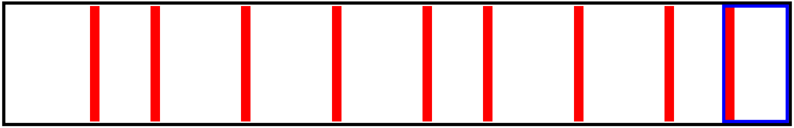
Algorithm: overview



Algorithm: overview

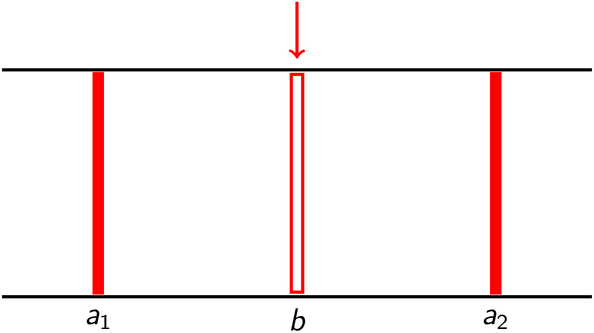


Algorithm: overview



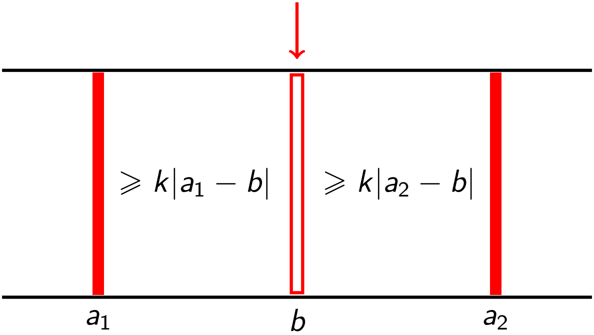
Inserting new separation

- The new separation cannot be 'close' to the neighbouring ones.



Inserting new separation

- The new separation cannot be 'close' to the neighbouring ones.
- There have to be at least $k|a_1 - b|$, $k|a_2 - b|$ vertices in between, respectively.



Subproblem

After guessing what exactly happens on neighbouring separators ($2^{O(k)}$ guesses), we have the following problem:

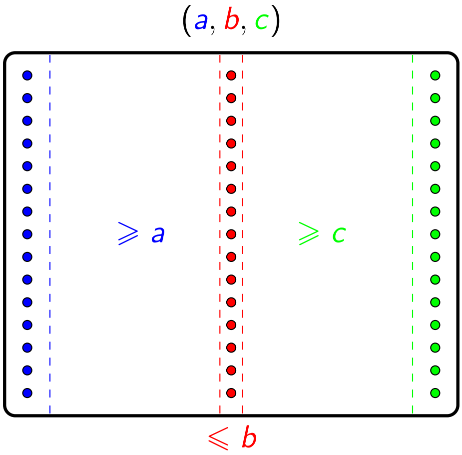
Tournament Balanced Separator

Input: A tournament S on n vertices; disjoint sets $X, Y \subseteq V(S)$; integers a, b, c.

Question: Does there exist a separation (C, D) of S such that

- $|C \cap D| \leqslant b$;
- $X \subseteq C \setminus D$, $Y \subseteq D \setminus C$;
- $|(C \setminus D) \setminus X| \geqslant a$ and $|(D \setminus C) \setminus Y| \geqslant c$?

Subproblem



Finding balanced separator

- Original implementation via brute-force enumeration of all $O(n^b)$ candidate separators.

Finding balanced separator

- Original implementation via brute-force enumeration of all $O(n^b)$ candidate separators.
- We show an algorithm working in time

$$2^{O(\min(a+c,b)\log(a+b+c))} \cdot n^2 \log n.$$

Finding balanced separator

- Original implementation via brute-force enumeration of all $O(n^b)$ candidate separators.
- We show an algorithm working in time

$$2^{O(\min(a+c,b)\log(a+b+c))} \cdot n^2 \log n.$$

- As $a, c = O(k^2)$ and $b \leqslant k$, this gives $2^{O(k \log k)} \cdot n^2 \log n$ for inserting one separation.

Finding balanced separator

- Original implementation via brute-force enumeration of all $O(n^b)$ candidate separators.
- We show an algorithm working in time

$$2^{O(\min(a+c,b)\log(a+b+c))} \cdot n^2 \log n.$$

- As $a, c = O(k^2)$ and $b \leqslant k$, this gives $2^{O(k \log k)} \cdot n^2 \log n$ for inserting one separation.
- Now we present a randomized version; derandomization via splitters.

Finding balanced separator

- Original implementation via brute-force enumeration of all $O(n^b)$ candidate separators.
- We show an algorithm working in time

$$2^{O(\min(a+c,b)\log(a+b+c))} \cdot n^2 \log n.$$

- As $a, c = O(k^2)$ and $b \leqslant k$, this gives $2^{O(k \log k)} \cdot n^2 \log n$ for inserting one separation.
- Now we present a randomized version; derandomization via splitters.
- Assume that a solution exists and fix one solution (C, D).

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:
  - $C \cap D$ get black,

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:
  - $C \cap D$ get black,
  - at least a nonterminals from $C \setminus (X \cup D)$ get white,

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:
  - $C \cap D$ get black,
  - at least a nonterminals from $C \setminus (X \cup D)$ get white,
  - at least c nonterminals from $D \setminus (Y \cup C)$ get white.

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:
  - $C \cap D$ get black,
  - at least a nonterminals from $C \setminus (X \cup D)$ get white,
  - at least c nonterminals from $D \setminus (Y \cup C)$ get white.
- **Probability**: at least $2^{-(a+b+c)}$.

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:
  - $C \cap D$ get black,
  - at least a nonterminals from $C \setminus (X \cup D)$ get white,
  - at least c nonterminals from $D \setminus (Y \cup C)$ get white.
- **Probability**: at least $2^{-(a+b+c)}$.
- By tweaking $1/2$ we get $2^{-O(\min(a+c,b)\log(a+b+c))}$.

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:
  - $C \cap D$ get black,
  - at least a nonterminals from $C \setminus (X \cup D)$ get white,
  - at least c nonterminals from $D \setminus (Y \cup C)$ get white.
- **Probability**: at least $2^{-(a+b+c)}$.
- By tweaking $1/2$ we get $2^{-O(\min(a+c,b)\log(a+b+c))}$.
- By repeating the experiment $2^{O(\min(a+c,b)\log(a+b+c))}$ times, with constant probability we hit the event.

Color coding

- Independently at random color every nonterminal white or black, with probability $1/2$.
- Examine the event:
  - $C \cap D$ get black,
  - at least a nonterminals from $C \setminus (X \cup D)$ get white,
  - at least c nonterminals from $D \setminus (Y \cup C)$ get white.
- **Probability**: at least $2^{-(a+b+c)}$.
- By tweaking $1/2$ we get $2^{-O(\min(a+c,b)\log(a+b+c))}$.
- By repeating the experiment $2^{O(\min(a+c,b)\log(a+b+c))}$ times, with constant probability we hit the event.
- Finding a solution respecting the coloring is polynomial time solvable.

Corollaries

- Corollaries of FPT approximation of pathwidth of a tournament:

Corollaries

- Corollaries of FPT approximation of pathwidth of a tournament:
  - Testing topological subgraph containment is FPT.

Corollaries

- Corollaries of FPT approximation of pathwidth of a tournament:
  - Testing topological subgraph containment is FPT.
  - Computing vertex deletion distance to any immersion-closed class of tournaments is FPT.

Corollaries

- Corollaries of FPT approximation of pathwidth of a tournament:
  - Testing topological subgraph containment is FPT.
  - Computing vertex deletion distance to any immersion-closed class of tournaments is FPT.
  - Follows from the fact that immersion relation is a well-quasi order on tournaments [Chudnovsky, Seymour].

More corollaries

- Testing rooted immersion in tournaments is FPT.

More corollaries

- Testing rooted immersion in tournaments is FPT.
- **Rooted**: some vertices of H may have prescribed images.

More corollaries

- Testing rooted immersion in tournaments is FPT.
- **Rooted**: some vertices of H may have prescribed images.
- Generalizes Edge-disjoint Paths.

More corollaries

- Testing rooted immersion in tournaments is FPT.
- **Rooted**: some vertices of H may have prescribed images.
- Generalizes Edge-disjoint Paths.
- We need additional irrelevant vertex technique in a triple.

More corollaries

- Testing rooted immersion in tournaments is FPT.
- **Rooted**: some vertices of H may have prescribed images.
- Generalizes Edge-disjoint Paths.
- We need additional irrelevant vertex technique in a triple.
  - Quite technical.

More corollaries

- Testing rooted immersion in tournaments is FPT.
- **Rooted**: some vertices of H may have prescribed images.
- Generalizes Edge-disjoint Paths.
- We need additional irrelevant vertex technique in a triple.
  - Quite technical.
- FPT was already known for closely related Rooted Infusion.

Later results

- P., *Computing cutwidth and pathwidth of semi-complete digraphs via degree orderings*, STACS 2013

Later results

- P., *Computing cutwidth and pathwidth of semi-complete digraphs via degree orderings*, STACS 2013
- Completely new approach.

Later results

- P., *Computing cutwidth and pathwidth of semi-complete digraphs via degree orderings*, STACS 2013
- Completely new approach.
- 7-approximation of pathwidth in $O(kn^2)$ time, instead of $O(OPT)$-approximation in $2^{O(k \log k)} \cdot n^3 \log n$ time.

Later results

- P., *Computing cutwidth and pathwidth of semi-complete digraphs via degree orderings*, STACS 2013
- Completely new approach.
- 7-approximation of pathwidth in $O(kn^2)$ time, instead of $O(OPT)$-approximation in $2^{O(k \log k)} \cdot n^3 \log n$ time.
- Running time of topological subgraph containment testing trimmed to $2^{O(|H| \log |H|)} \cdot n^2$.

Conclusions

- FPT approximation of pathwidth opens possibilities for new FPT results on tournaments.

Conclusions

- FPT approximation of pathwidth opens possibilities for new FPT results on tournaments.
- **Open problem**: VERTEX-DISJOINT PATHS

Conclusions

- FPT approximation of pathwidth opens possibilities for new FPT results on tournaments.
- **Open problem**: Vertex-disjoint Paths
  - k terminal pairs: $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.

Conclusions

- FPT approximation of pathwidth opens possibilities for new FPT results on tournaments.
- **Open problem**: Vertex-disjoint Paths
  - k terminal pairs: $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.
  - Can one find vertex-disjoint paths $P_1, P_2, \ldots, P_k$ connecting corresponding terminals?

Conclusions

- FPT approximation of pathwidth opens possibilities for new FPT results on tournaments.
- **Open problem**: Vertex-disjoint Paths
  - k terminal pairs: $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.
  - Can one find vertex-disjoint paths $P_1, P_2, \ldots, P_k$ connecting corresponding terminals?
- The problem is known to be in XP by a different approach [Chudnovsky, Scott, Seymour]. FPT is not known.

Conclusions

- FPT approximation of pathwidth opens possibilities for new FPT results on tournaments.
- **Open problem**: VERTEX-DISJOINT PATHS
  - k terminal pairs: $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.
  - Can one find vertex-disjoint paths $P_1, P_2, \ldots, P_k$ connecting corresponding terminals?
- The problem is known to be in XP by a different approach [Chudnovsky, Scott, Seymour]. FPT is not known.
- The current technique fails because of the irrelevant vertex rule.

Thank you

Questions?