Topological problems in tournaments

Michał Pilipczuk

Department of Informatics, University of Bergen, Norway

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Undirected graphs

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- FPT algorithm for minor testing.
- **Measure of complexity**: treewidth.

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- Problems at each step.
- No natural containment notion (topological minor?, immersion?)
- Containment problems are NP-hard for small, constant size graphs H [Fortune et al.]
- No natural analogue of treewidth.

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- The class of **tournaments** has been identified as such by Chudnovsky, Fradkin, Kim, Scott, and Seymour.
  - The results hold also for a sligthly more general class of *semi-complete* digraphs.

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- **Goal**: make an overview of the topic.

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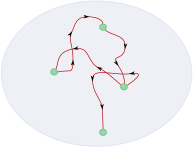
- **Semi-complete digraph**: $\forall v, w \in V(T)$, **at least** one of (v, w) and (w, v) belongs to $E(T)$.

Containment notions

Immersion
vertices map to vertices,
arcs to arc-disjoint paths



Topological subgraph
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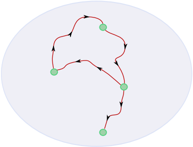

Minor
vertices map to vertex-disjoint strongly
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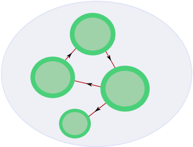

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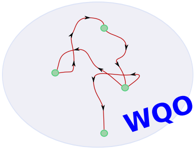

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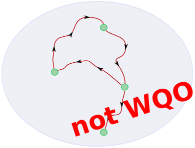

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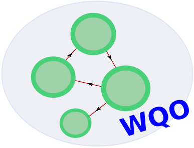
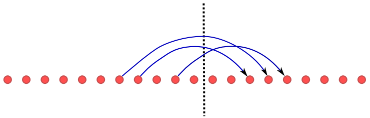


Containment notions vs. width parameters

Immersion	**Cutwidth** 
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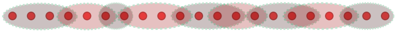
Minor	

Decomposition-obstacle theorems

Cutwidth vs. immersion

If T does not contain H as an immersion, then $\mathbf{ctw}(T) \leqslant f(|H|)$.

Pathwidth vs. top. subgraph/minor

If T does not contain H as a topological subgraph/minor, then $\mathbf{pw}(T) \leqslant f(|H|)$.

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If T does not contain H as a topological subgraph/minor, then $\mathbf{pw}(T) \leqslant f(|H|)$.

- **Important**: the proofs prove decomposition-obstacle theorems
- that can be turned into approximation algorithms: the algorithm either gives a decomposition, or an appropriate combinatorial obstacle.

Overview of algorithms

Problem	CFKSS	FP, SODA 2013	P, STACS 2013	FP, arxiv
Cutwidth approx.	$O(n^3)$ (width $O(k^2)$)			
Cutwidth exact	$f(k) \cdot n^3$, non-u, non-c			
Pathwidth approx.	$n^{f(k)}$ time (width $O(k^2)$)			
Pathwidth exact	$n^{f(k)}$ non-u, non-c			
Immersion test	$f(\|H\|) \cdot n^3$			
T. subgraph/minor test	$n^{f(\|H\|)}$			

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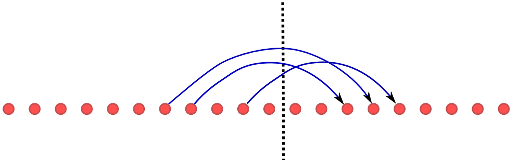

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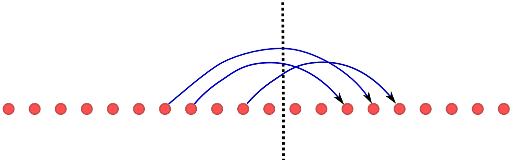

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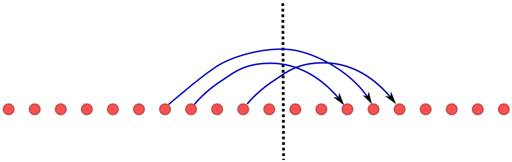

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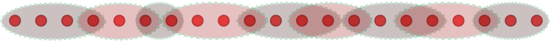

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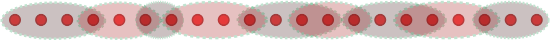

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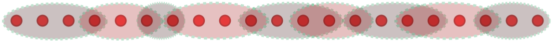

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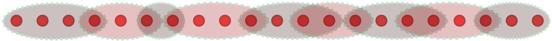

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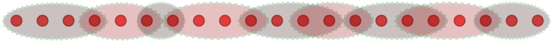

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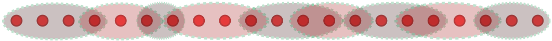

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- Width of $\mathcal{P}$ is $\max |W_i| - 1$.
- Pathwidth of T, denoted $\mathbf{pw}(T)$, is minimum width among the path decompositions of T.

The degree ordering approach

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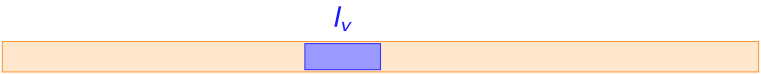

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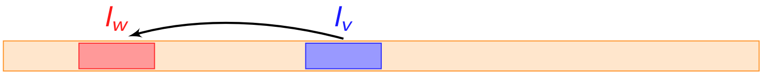

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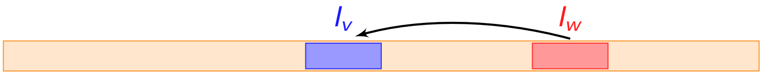

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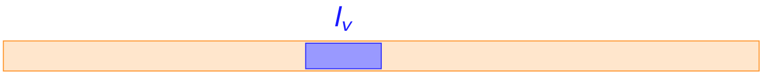

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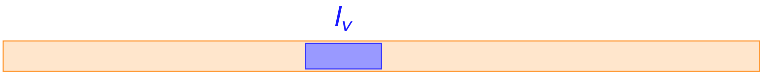

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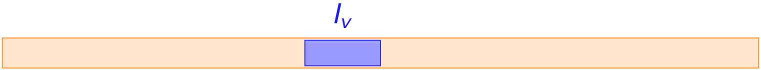

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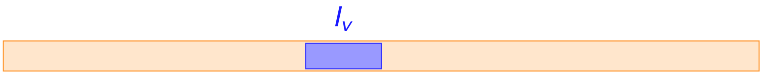

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  - Possible aberrations due to vertices sharing bags, but their number is limited by the width of the decomposition.
- Ordering of vertices by the outdegree should give a crude idea of how the optimum decomposition looks.
- In particular, there should not be many vertices with similar outdegrees.

Obstacles: approach of Fradkin and Seymour

Run the algorithm

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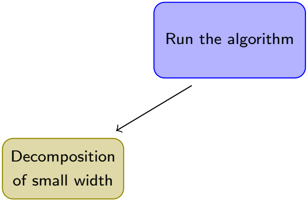

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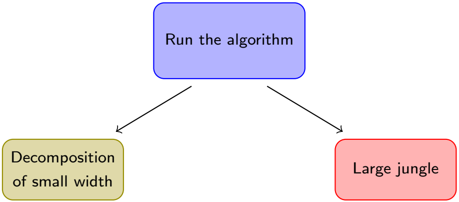

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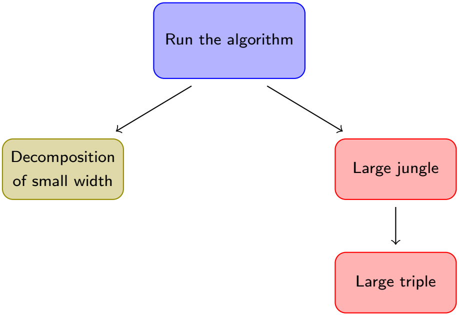

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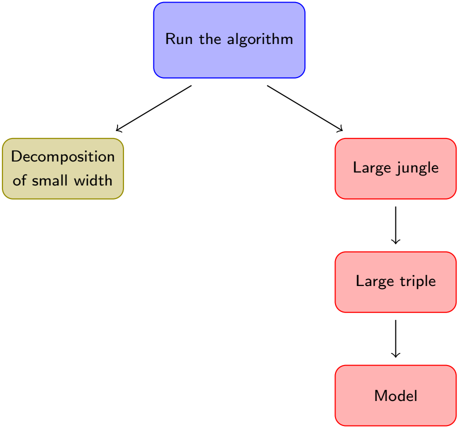

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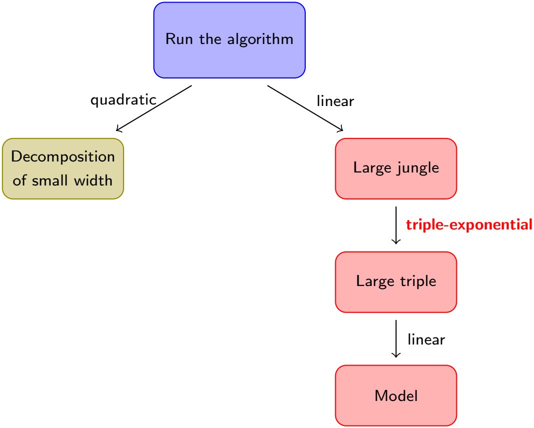

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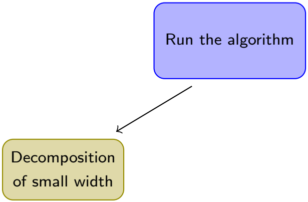

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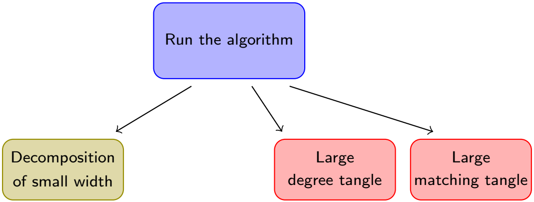

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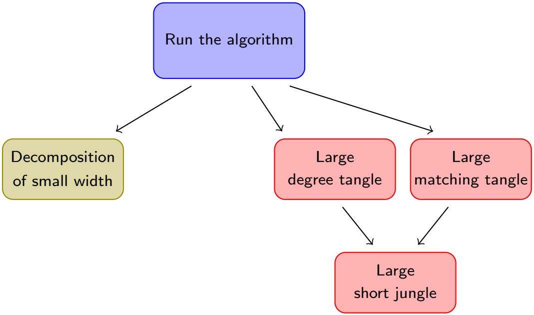

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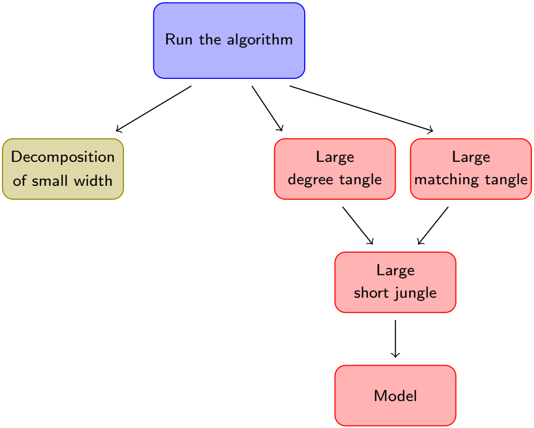

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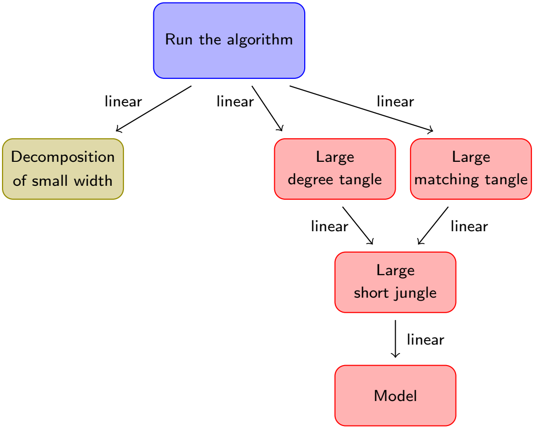

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- **Open problem**: find better obstacles, on which irrelevant vertex rules can be employed.

Open problem #2

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- **Open problem**: Is it possible to develop a meta-explanation of subexponential-time FPT algorithms on tournaments, similarly to bidimensionality?

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Question: Is there a family of vertex-disjoint paths $P_1, P_2, \ldots, P_k$, s.t. P_i leads from s_i to t_i?

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- Problem with the current approach: irrelevant vertex rule on a triple fails.

Open problems

- Finding better obstacles for irrelevant vertex rules.
- Bidimensionality for tournaments?
- FPT algorithm for vertex-disjoint paths.