Everything you always wanted to know about the parameterized complexity of Subgraph Isomorphism

(but were afraid to ask)

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  - Trivial $n^{\mathcal{O}(k)}$ algorithm.
  - Generalization of Clique $\Rightarrow$ no $f(k) \cdot n^{\mathcal{O}(1)}$ algorithm (FPT), unless FPT=W[1].

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- **Question**: Could we get something better if we assume that $\mathrm{tw}(G)$ is small, instead of $\mathrm{tw}(H)$?

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  - assuming that H is connected.

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- **Our motivation**:
  - **It's a mess.**
  - Let's try to clean it.

Parameters

We consider the following 10 parameters for H and G:

1. Number of vertices $|V(\cdot)|$ (only H).
2. Number of connected components $\mathrm{cc}(\cdot)$.
3. Maximum degree $\Delta(\cdot)$.
4. Treewidth $\mathrm{tw}(\cdot)$.
5. Pathwidth $\mathrm{pw}(\cdot)$.
6. Feedback vertex set number $\mathrm{fvs}(\cdot)$.
7. Cliquewidth $\mathrm{cw}(\cdot)$.
8. Genus $\mathrm{genus}(\cdot)$.
9. Hadwiger number (largest clique minor) $\mathrm{hadw}(\cdot)$.
10. Topological Hadwiger number (largest topological clique minor) $\mathrm{hadw}_T(\cdot)$.

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5 possible additional restrictions on H or G:

1. Genus is 0 (i.e., planar).
2. Number of components is 1 (i.e., connected).
3. Treewidth is at most 1 (i.e., graph is a forest).
4. Maximum degree at most 2 (i.e., paths and cycles).
5. Maximum degree at most 3.

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Determine for every combination of these parameters and constraints, whether there is an algorithm with running time

$$f_1(p_1, p_2, \ldots, p_\ell) \cdot n^{f_2(p_{\ell+1}, \ldots, p_t)},$$

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- Vectors $(p_1, \ldots, p_\ell)$, $(p_{\ell+1}, \ldots, p_t)$, $(c_1, \ldots, c_q)$, are called the *description*.

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- Thus we obtain a **partial order** of descriptions.
- There are $> 3^{19}$ descriptions, but maybe a smaller set of results explains all of them.

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- All the cases are explained by 11 maximally-positive and 17 minimally-negative results.
- Out of these, 4 algorithmic results are new: they are contained in 2 new families of tractable cases of Subgraph Isomorphism.
- 11 negative results are new: a new methodology for proving intractability on planar and bounded-genus graphs.

Results

Short Description	Thm	H										G								
		$\lvert V(\cdot)\rvert$	cc	Δ	fvs	pw	tw	cw	genus	hadw	hadw_T	cc	Δ	fvs	pw	tw	cw	genus	hadw	hadw_T
FO model checking	Thm P.1 (page 17)	M															M			
	Thm P.2 (page 17)	M																		M
Color coding	Thm P.3 (page 17)	M					E													
Matoušek-Thomas	Thm P.4 (page 18)		M	M												E				
Paths&Cycles $\to$ Paths&Cycles	Thm P.5 (page 18)											E	2							
Dynamic Programming	Thm P.6 (page 19)		E	2												M				
	Thm P.7 (page 21)		E	2													E			
	Thm P.8* (page 23)		M													1				
FVS and CSPs	Thm P.9* (page 28)		M	2									M	M						
	Thm P.10* (page 36)		E										M	M				E		
	Thm P.11* (page 46)		E	E									M	M					E	
Bin Packing	Thm N.1 (page 46)											M	2			1				
	Thm N.2 (page 47)		1				1							E	E			0		
	Thm N.3 (page 47)			2								1	3		E	1				
Planar cubic HamPath	Thm N.4 (page 48)		1	2			1						3					0		
Clique	Thm N.5 (page 48)	M	1					E												
HamPath in bounded **cw**	Thm N.6 (page 48)		1	2			1										M			
Grid Tiling, 1-in-n gadgets	Thm N.7* (page 57)		M			E	1					1	3	M	M			0		
	Thm N.8* (page 61)		1			E	1						M	M	M			M	E	
	Thm N.9* (page 61)		1			E	1						3	M	M			M		
	Thm N.10* (page 63)		1	3		E	1						M	M	M		E	M		
Grid Tiling, moustache gadgets	Thm N.11* (page 64)		1	3		E	1							M	M			0		
	Thm N.12* (page 66)		1			E	1						3		M			0		
Small planar graph	Thm N.13* (page 67)	M	1	3					0											
Exact Planar Arc Supply	Thm N.14* (page 74)		M	2			1					1		M	M			0		
	Thm N.15* (page 77)		M	2			1					1	3		M			0		
	Thm N.16* (page 79)		M	2			1					1		M	M		E	M		
	Thm N.17* (page 79)		M	2			1					1	M		M		E	M		

Figure 1: Positive and negative results in the paper. Results marked with * are new findings that were not known before.

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  - Source code and input files available as ancillary files in the arxiv version.

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- Take one vertex and guess its image; now we are at subgraph isomorphism of *rooted trees*.
- Create dynamic programming table $D[u][v]$, for $u \in V(H)$ and $v \in V(G)$, with the following meaning:

$$D[u][v] = \text{Is there a subgraph isomorphism } \eta \text{ from } H_u \text{ to } G_v \text{ such that } \eta(u) = v?$$

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- Result is $D[r_H][r_G]$.

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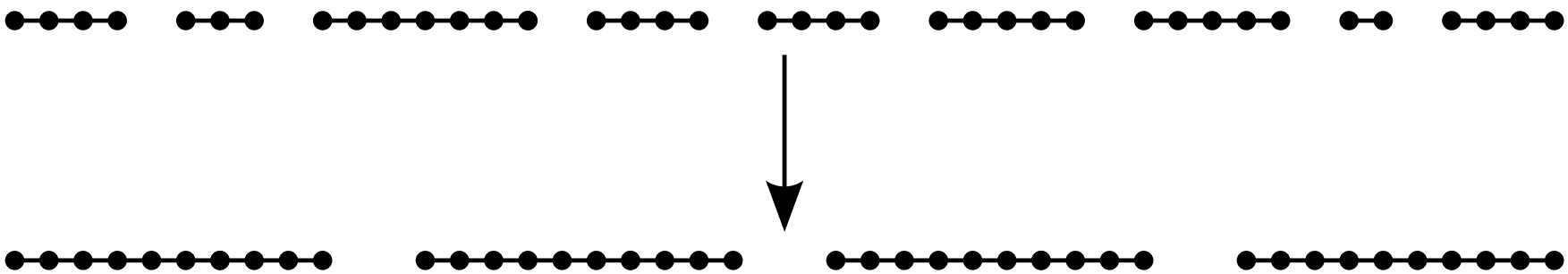
- Bin Packing: Subgraph Isomorphism is already NP-hard for forests of paths!



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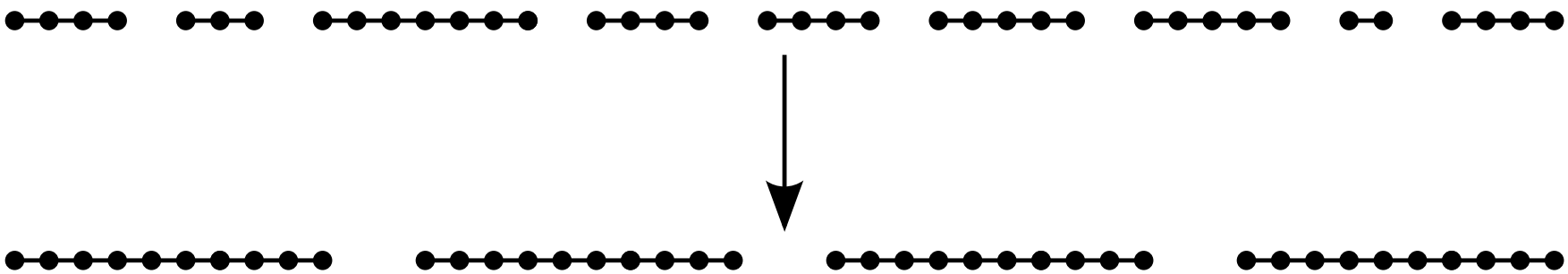

- We show an $f(\mathrm{cc}(H)) \cdot n^{\mathcal{O}(1)}$ algorithm for packing a forest into a forest.

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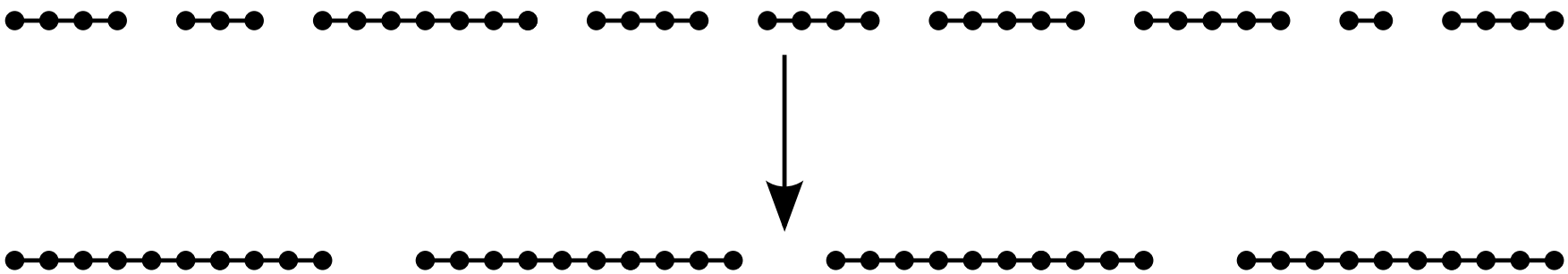

- We show an $f(\mathrm{cc}(H)) \cdot n^{\mathcal{O}(1)}$ algorithm for packing a forest into a forest.
- A very natural case omitted by the previous study (Thm P.8).

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  - a subset S of the other components of H.
- **Meaning**: $D[u][v][S]$ is true iff the subgraph $H_u \cup S$ can be squeezed into G_v so that u is mapped to v.

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- Computation of $D[u][v][S]$ involves a variant of maximum matching with $\leq k$ colors.
- This variant is NP-hard, but can be solved in FPT time when parameterized by k using algebraic algorithms for matchings.
- The algorithm uses Schwartz-Zippel lemma. Hence it is randomized, and we do not know how to derandomize it.

Feedback vertex set number and planarity

- By Matoušek and Thomas, Subgraph Isomorphism can be solved in time $f(\Delta(H)) \cdot n^{\mathcal{O}(\mathrm{tw}(G))}$ for connected H.

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- **Recall**: X is a feedback vertex set if $G \setminus X$ is a forest. $\mathrm{fvs}(G)$ is the minimum size of a fvs in G.

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Main algorithmic result, Thm P.10

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- **Lower bounds**: all these assumptions are necessary!

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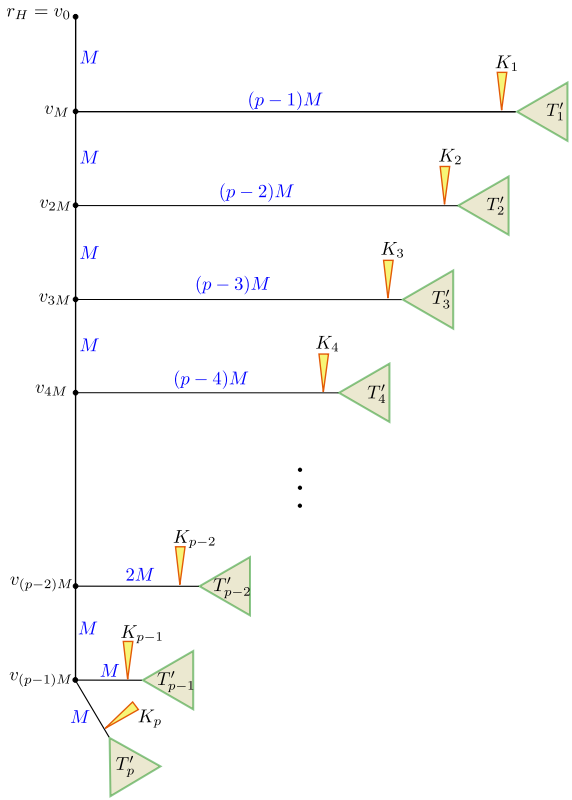

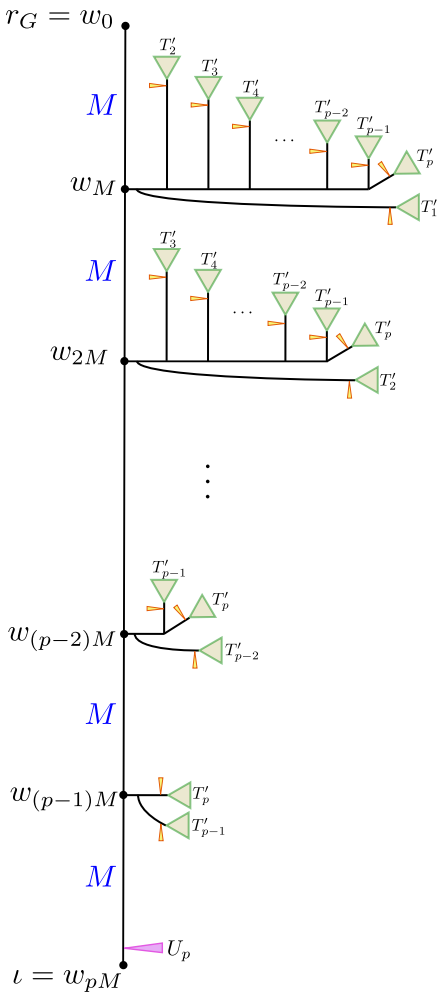

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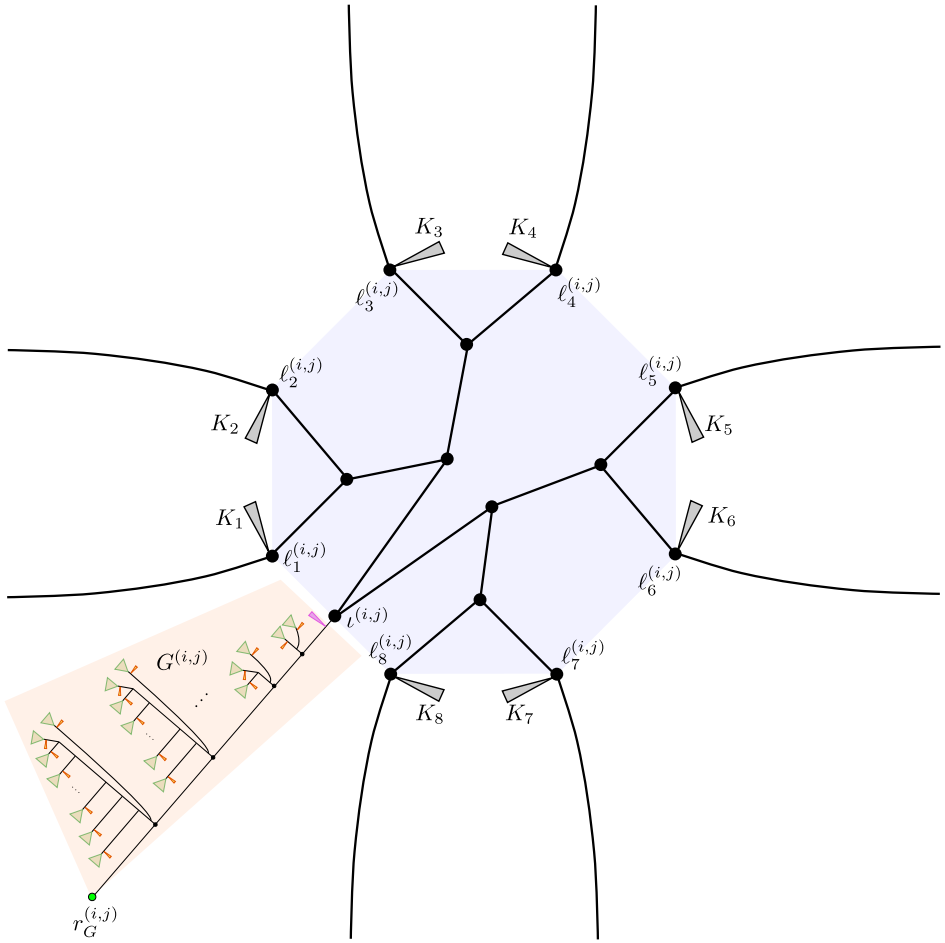

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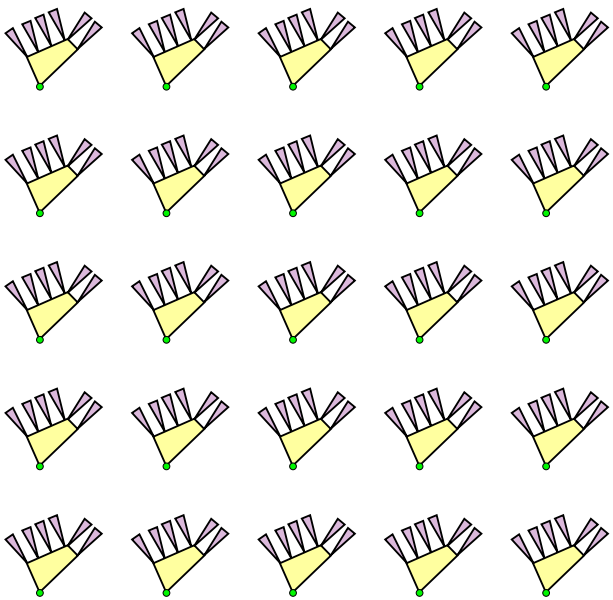

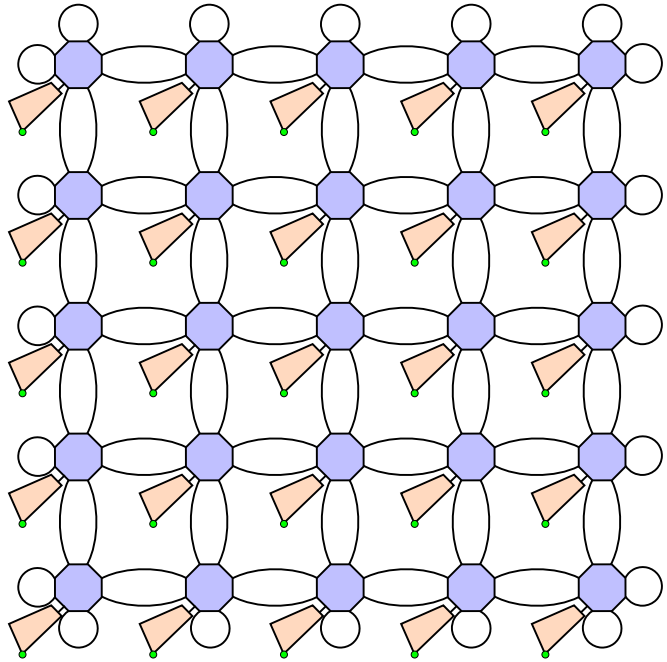

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- **Thanks for attention!**