Minimum Bisection is fixed-parameter tractable

Marek Cygan[1], Daniel Lokshtanov[2], Marcin Pilipczuk[2],
Michał Pilipczuk[2], Saket Saurabh[2,3]

[1] Institute of Informatics, University of Warsaw, Poland

[2] Department of Informatics, University of Bergen, Norway

[3] Institute of Mathematical Sciences, Chennai, India

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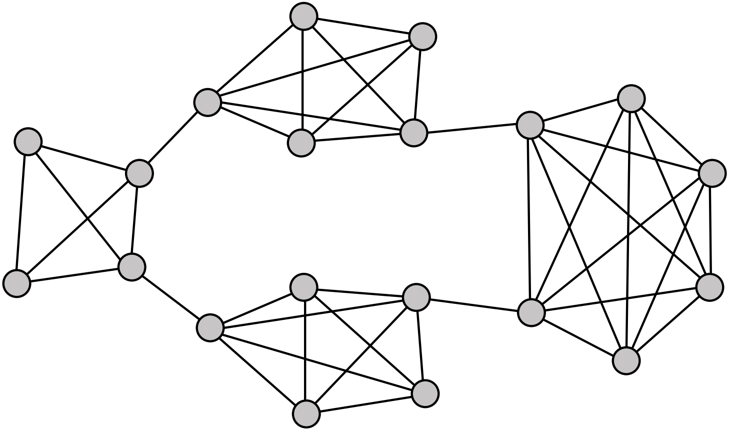
The problem

Mininimum Bisection

Input: A graph G with $|V(G)| = 2n$ and an integer k

Question: Does there exist a partition (A, B) of $V(G)$ such that $|A| = |B| = n$ and $|E(A, B)| \leq k$?

Example



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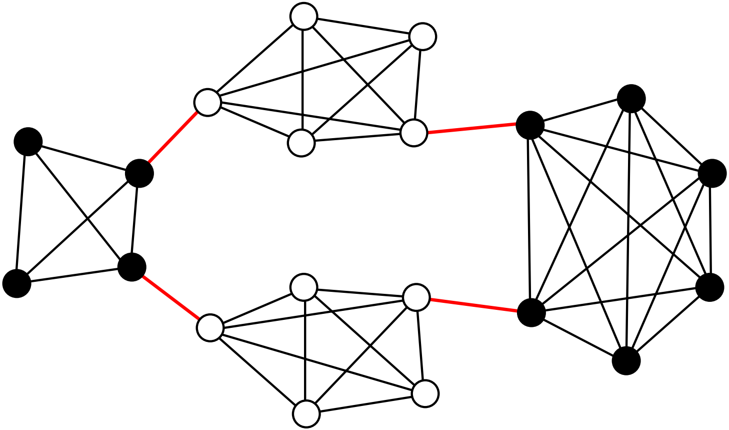

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  - It is open whether MinBisection is NP-hard on planar graphs.
- **Our result**: MinBisection is fixed-parameter tractable when parameterized by k.
- **Running time**: $2^{\mathcal{O}(k^3)} \cdot n^3 \log^3 n$

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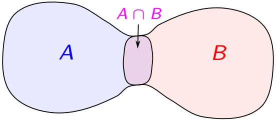
Unbreakable sets

- (A, B) is a *separation* if $A \cup B = V(G)$ and $E(A \setminus B, B \setminus A) = \emptyset$. $A \cap B$ is the *separator*, and $|A \cap B|$ is the *order* of the separation.



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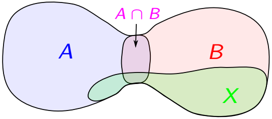

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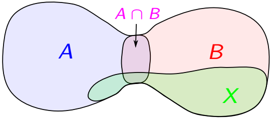
- **Intuition**: a separation of small order can carve out only a small portion of X.



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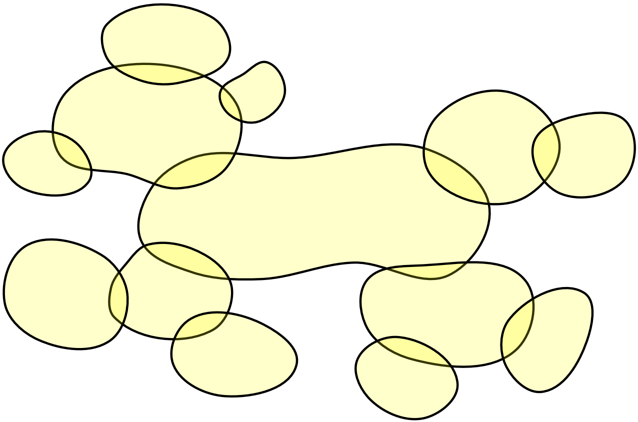
Moreover, this decomposition can be computed in $2^{\mathcal{O}(k^2)} \cdot n^2 m$ time.

On a picture



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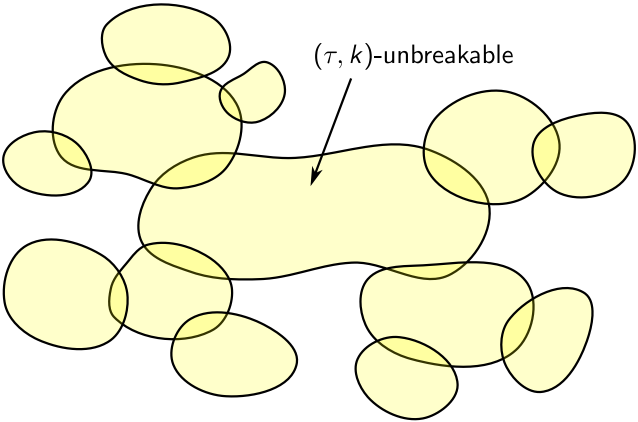

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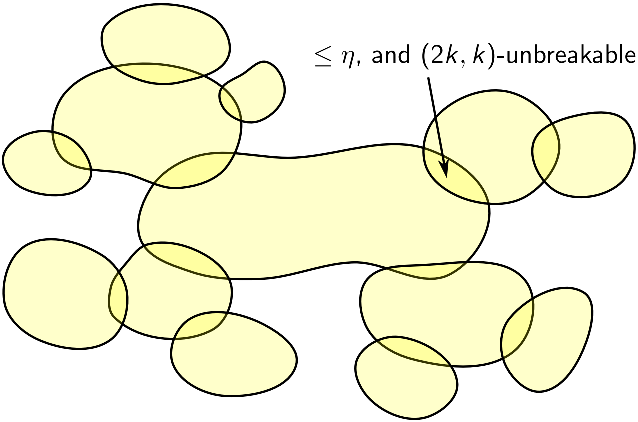

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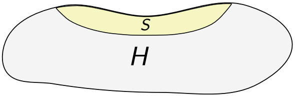

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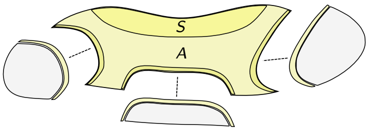

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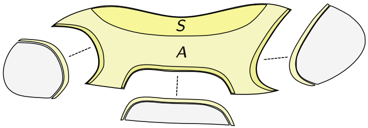
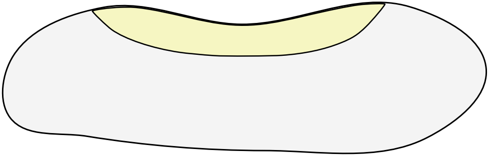
- Then recurse into the connected components below.



Breakable S

- Test whether S is (k, k)-breakable in H.



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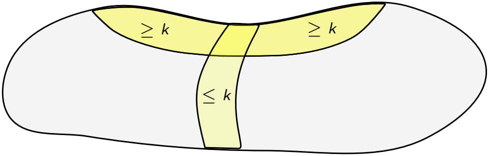

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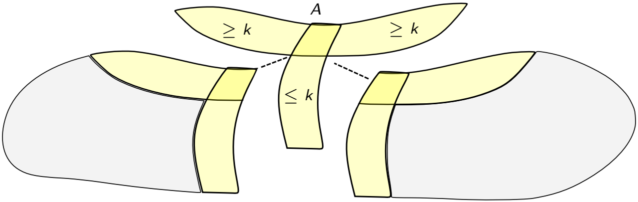

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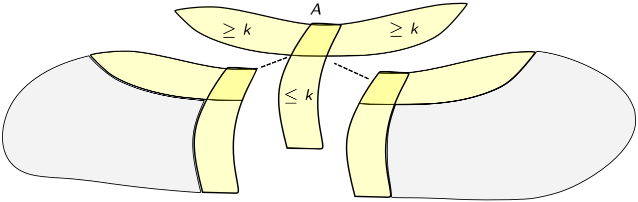
- Every connected component below is either on the left or on the right, so it sees only $\leq \eta$ vertices from A.



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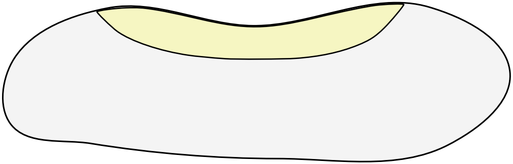

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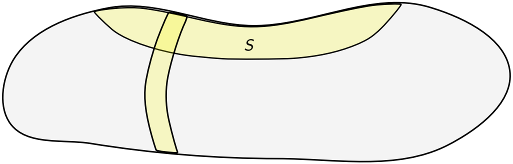

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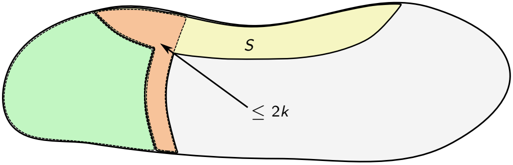

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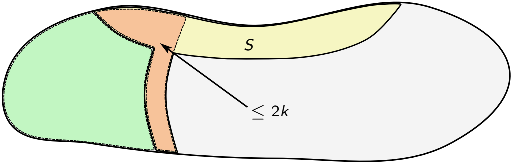
- **Idea**: Mark greedily **all** possible such carvings; what is left constitutes A.



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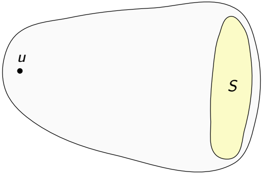

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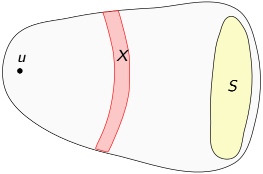

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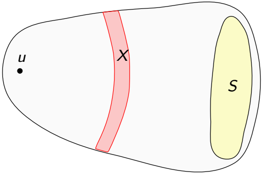

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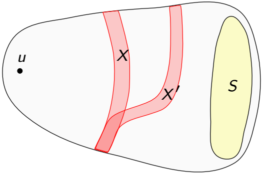

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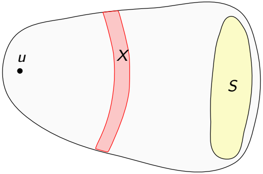

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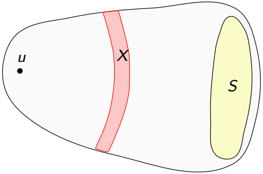

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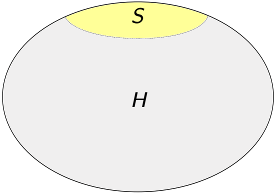
- **Note**: minimality $\Rightarrow X = N(\text{reach}(u, H \setminus X))$.
- For fixed u and S, there are at most 4^k important u-S separators of size at most k. [Chen et al., Marx and Razgon]

Chips and the decomposition



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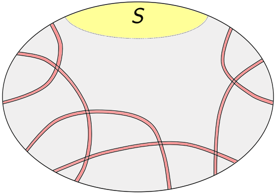

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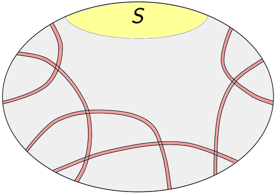

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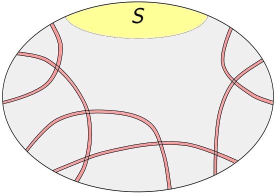

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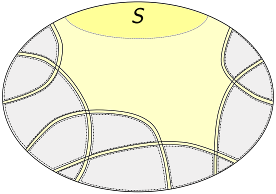

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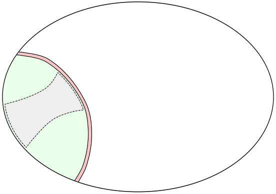
- Set $A = \bigcup_{C \in \mathcal{C}} N(C) \cup \bigcap_{C \in \mathcal{C}} V(H) \setminus N[C]$.

Bounding the adhesions



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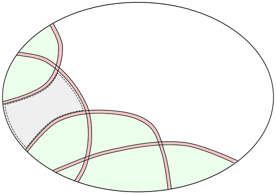

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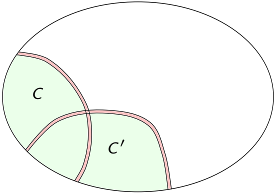

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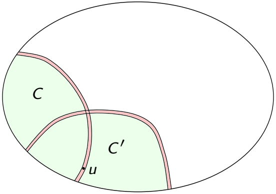

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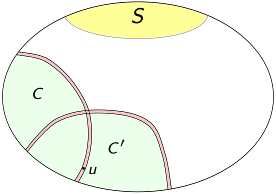
- Connectivity $\Rightarrow$ There exists $u \in N(C) \cap C'$.

Bounding the adhesions, continued



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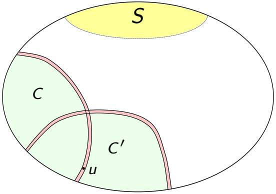

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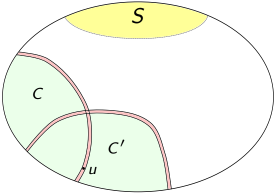

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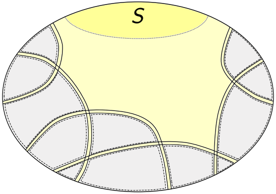

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- **Check**: A is (τ, k)-unbreakable in H, for some $\tau = 2^{\mathcal{O}(k)}$.

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- **Thanks for attention!**