Sparsity — tutorial 7

Low treedepth colorings

Definition 0.1. A *tree-decomposition* of a graph G is a pair (T, β) consisting of a tree T and a function $\beta : V(T) \to 2^{V(G)}$ such that for all $v \in V(G)$ the set $\{t \in V(T) \mid v \in \beta(t)\}$ is non-empty and induces a subtree of T and for every edge $e \in E(G)$ there is $t \in V(T)$ with $e \subseteq \beta(t)$.

Problem 1. Let G be an n-vertex graph and let (T, β) be a tree decomposition of G with all bags of size at most w. Prove that $\mathtt{td}(G) \leq w \cdot \mathtt{td}(T)$.

Problem 2. Given a graph G and its tree decomposition (T, β) with all bags of size at most w, show that one can test if G is 3-colorable in time $\mathcal{O}(3^w \cdot w^2 \cdot |V(G)|)$.

Problem 3. Prove that there exists an algorithm that, given an n-vertex graph G together with its tree-depth decomposition of height at most d, verifies whether G admits a proper 3-coloring in time $\mathcal{O}(3^d \cdot n^c)$ and space $\mathcal{O}(n^c)$, for some constant c independent of d. The constants hidden in the $\mathcal{O}(\cdot)$-notation may **not** depend on d.

Problem 4. Given a graph H, a graph G and its tree decomposition (T, β) with all bags of size at most w, show that one can in $f(H, w) \cdot |V(G)|$ time check if H is isomorphic to a subgraph of G for some computable function H.

Show that one can also count the number of subgraphs isomorphic to H within the same time bound.

- A *centered coloring* of a graph G is a function $f : V(G) \to \mathbb{N}$ such that every connected subgraph H of G has a vertex of unique color in H.
- A *linear coloring* of a graph G is a function $f : V(G) \to \mathbb{N}$ such that every simple path of G has a vertex of unique color.
- A *p-treedepth coloring* is a function $f : V(G) \to \mathbb{N}$ such that every connected subgraph H of G that uses $i \leq p$ colors has treedepth at most i.
- A *p-centered coloring* is a function $f : V(G) \to \mathbb{N}$ such that every connected subgraph H of G has a vertex of unique color in H or uses more than p colors.
- A *p-linear coloring* is a function $f : V(G) \to \mathbb{N}$ such that every simple path in G has a vertex of unique color or uses more than p colors.

Problem 5. Prove that if a graph admits a p-treedepth coloring with M colors, then it also admits a p-centered coloring with $M \cdot p^{\binom{M}{<p}}$ colors.

Problem 6. Show an example of a graph where the minimum number of colors required for a centered coloring is strictly larger than the minimum number of colors required for a linear coloring. Can you make your example a tree?

Problem 7. Show that for every $\varepsilon > 0$ there exists a graph G and an integer k such that G admits a linear coloring with k colors but any centered coloring of G requires at least $(2 - \varepsilon)k$ colors.

Problem 8. Prove that any graph of treedepth at most k can be colored with at most 2^k colors with the following property: for any two vertices of the same color, the distance between the two vertices in question is even.

Problem 9. For a graph G and $r \in \mathbb{N}$, by $G^{=r}$ we denote the graph on vertex set $V(G)$ where two vertices u and v are adjacent if and only if the distance between them in G is equal *exactly* to r.

Prove that for every odd integer $r \in \mathbb{N}$ and class of bounded expansion $\mathcal{C}$, there exists a number M such that for every $G \in \mathcal{C}$, the graph $G^{=r}$ admits a proper coloring with M colors.