Sparsity — tutorial 2

Bounded expansion and nowhere denseness

October 11th, 2019

Problem 1. Prove that for every positive integer k there exists a simple graph G that has maximum degree at most k and girth at least k, but contains at least $\frac{k|V(G)|}{4}$ edges.

Problem 2. Prove that the class $\mathcal{C} = \{G : \Delta(G) \leq \mathsf{girth}(G)\}$ is nowhere dense and has unbounded expansion.

Problem 3. Prove that for a graph class $\mathcal{C}$, the following conditions are equivalent:

- There is a constant $c \in \mathbb{N}$ such that $\nabla_d(\mathcal{C}) \leq c$ for all $d \in \mathbb{N}$.
- There is a graph H such that H is not a minor of any graph from $\mathcal{C}$.

Problem 4. Prove that for a graph class $\mathcal{C}$, the following conditions are equivalent:

- The class $\mathcal{C}$ is somewhere dense.
- There is $d \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, the d-subdivision of K_n is a subgraph of some graph from $\mathcal{C}$.

Problem 5. Prove that a k-degenerate n-vertex graph has at most $2^k \cdot n + 1$ cliques.

Problem 6. Suppose G is a graph and $A \subseteq V(G)$ some subset of its vertices. Define the following equivalence relation $\sim_A$ on the vertices of $V(G) - A$:

$$u \sim_A v \qquad \text{if and only} \qquad N[u] \cap A = N[v] \cap A.$$

Prove that

- in $V(G) - A$, the number of vertices with at least $2\nabla_0(G)$ neighbors in A is at most $|A|$; and
- $\sim_A$ has at most $\left(4^{\nabla_1(G)} + \nabla_1(G)\right) \cdot |A| + 1$ equivalence classes.

Problem 7. Prove that if G is a graph with $\nabla_0(G) \leq k$ and A is a subset of its vertices, then there exists a vertex subset $B \supseteq A$ such that $|B| \leq 2|A|$ and every vertex of $V(G) - B$ has at most $2k$ neighbors in B.

Problem 8. Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $d \in \mathbb{N}$ there exists a constant c_d such that the following holds. For every graph $G \in \mathcal{C}$ and every subset A of its vertices, there exists a vertex subset $B \supseteq A$ such that $|B| \leq c_d|A|$ and for every vertex $u \in V(G) - B$, at most c_d vertices of B can be reached from u by a path of length at most d whose internal vertices do not belong to B.

Problem 9. Prove that every clique can be obtained as an induced $(2, \infty)$-packing graph in a planar graph.

Problem 10. Suppose $\mathcal{F}$ is a family of closed Euclidean balls in $\mathbb{R}^k$, not necessarily of equal radii and not necessarily disjoint. The *ply* of $\mathcal{F}$ is the maximum number of balls that intersect at one point; that is, $\mathcal{F}$ has ply at most ρ iff every point in $\mathbb{R}^k$ is in at most ρ balls of $\mathcal{F}$. For $\rho, d \in \mathbb{N}$, let $\mathcal{B}_{\rho,k}$ be the class of intersection graphs of families of balls of ply at most ρ in $\mathbb{R}^k$; that is, $G \in \mathcal{B}_{\rho,k}$ if with every vertex of G we can associate a closed ball in $\mathbb{R}^d$ so that the balls form a family of ply at most ρ and two vertices are adjacent in G if and only if the corresponding balls intersect.

Prove that for all fixed $\rho, k \in \mathbb{N}$, there is a polynomial $p(\cdot)$ of degree k such that $\nabla_d(\mathcal{B}_{\rho,k}) \leq p(d)$ for all $d \in \mathbb{N}$.