Sparsity — tutorial 13

Polynomial expansion

Problem 1. Let G be a planar graph of radius r. Prove that G has tree-width at most $3r$.

Problem 2. Let G be an n-vertex planar graph. Prove that there exists a balanced separator of G of size at most $6\sqrt{n}$.

Problem 3. Prove that there is an (E)PTAS for the following problems on planar graphs:
Vertex Cover, Independent Set, Dominating Set, Distance-r Dominating Set.

Problem 4. Prove that if a class $\mathcal{C}$ has polynomial expansion, then there exists a constant $\delta > 0$ such that every n-vertex graph from $\mathcal{C}$ has tree-depth at most $\mathcal{O}(n^{1-\delta})$. Conclude that Independent Set on graphs from $\mathcal{C}$ can be solved in time $2^{\mathcal{O}(n^{1-\delta})}$.

Problem 5. Prove that for every class $\mathcal{C}$ of polynomial expansion there exists a polynomial $p(x)$ such that $\mathrm{adm}_r(G) \leq p(r)$ for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$.

Problem 6. Prove that every class $\mathcal{C}$ such that there is a polynomial $p(x)$ and $\mathrm{scol}_r(G) \leq p(r)$ for all $r \in \mathbb{N}$ has polynomial expansion.

Problem 7. For a graph G and an integer $t \geq 0$, let $G^{(t)}$ be the graph G with every edge subdivided t times (i.e., replaced with a path of length $t+1$). Consider the following class of graphs:

$$\mathcal{C} = \{G^{(6 \cdot \mathrm{treewidth}(G))} \mid G \text{ is a graph}\}.$$

- Prove that $\mathcal{C}$ is of polynomial expansion.
- Prove that there is no polynomial p such that $\mathrm{wcol}_r(G) \leq p(r)$ for every $r \geq 0$ and $G \in \mathcal{C}$.