ALGOMANET Sparsity
Tutorial 5: Sparsity and logic
January 24th, 2020

Problem 1. Let $\mathcal{C}$ be a class of bounded expansion and $d, k \in \mathbb{N}$ be fixed. Prove that the following problem can be solved in linear time on graphs from $\mathcal{C}$: given $G \in \mathcal{C}$ and a distance-d dominating set D in G, find another distance-d dominating set D' in G such that $|D'| < |D|$ and $|D \triangle D'| \leqslant k$, or conclude that no such D' exists.

Definition 1. For a graph G, vertex subset A, and FO formula $\varphi(x, y)$ in the signature of graphs, we define

$$S^{\varphi}(A) \coloneqq \{\, \{\, u \in A : \; G \models \varphi(u, v) \,\} : \; v \in V(G) \,\}.$$

Problem 2. Prove that for every graph class $\mathcal{C}$ of bounded expansion and FO formula $\varphi(x, y)$ in the signature of graphs, there exists a constant $c \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ and $A \subseteq V(G)$, we have

$$|S^{\varphi}(A)| \leqslant c \cdot |A|.$$

Problem 3. Prove that if $\mathcal{C}$ is a subgraph-closed somewhere dense class of graphs, then $\mathcal{C}$ is not stable.

Problem 4. *Map graphs* are intersection graphs of closed polygons in the plane with disjoint interiors. Prove that map graphs are stable.

Problem 5. Prove that for every graph class $\mathcal{C}$ and FO formula $\varphi(\bar{x}, \bar{y})$ in the signature of graphs, φ has a finite semi-ladder index on $\mathcal{C}$ if and only if it has a finite ladder index and a finite co-matching index on $\mathcal{C}$.

Definition 2. We say that a class of graphs $\mathcal{C}$ has the *Erdős-Hajnal property* if there exists $\delta > 0$ such that every n-vertex graph $G \in \mathcal{C}$ either contains a clique or an independent set of size at least n^{δ}.

In the following few problems, we will prove the following theorem.

Theorem 1. *For every nowhere dense class of graphs $\mathcal{C}$ and symmetric FO formula $\varphi(x, y)$ in the signature of graphs, the class $\varphi(\mathcal{C})$ has the Erdős-Hajnal property.*

In the following, a binary tree is a prefix-closed finite subset of $\{\mathrm{D}, \mathrm{S}\}^{\star}$.

Consider constructing a binary tree labelled with all the elements of $V(G)$, by starting from an empty tree and inserting vertices one by one. When inserting a vertex v, we first compare it to the root u using the following question: does $G \models \varphi(u, v)$? If the answer is positive, proceed to the son of u, otherwise proceed to the daughter of u. We continue proceeding down the tree choosing directions using the question above until we reach a leaf that is currently unoccupied. Then v is put at this leaf.

Let τ be the obtained tree.

Problem 6. Prove that if τ has a node with k letters D, then $\varphi(G)$ contains an independent set of size k, and if τ has a node with k letters S, then $\varphi(G)$ contains a clique of size k.

Problem 7. Prove that if τ has a node in which the blocks of letters D and S alternate at least $2t$ times, then G contains a φ-ladder of length t.

Problem 8. Conclude the proof of Theorem 1 using the problems above.