ALGOMANET Sparsity

Tutorial 4: Uniform quasi-wideness

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Problem 1. Prove that a class of graphs is uniformly wide if and only if it has bounded maximum degree.

Problem 2. Prove that every uniformly quasi-wide class is nowhere dense.

Definition 1. A set of vertices I in a graph G is *distance-d independent* if vertices of I are pairwise at distance more than d. A set of vertices D is *distance-d dominating* if every vertex of G is at distance at most d from a vertex of D. The size of a largest distance-d independent set in G is denoted by $\mathrm{ind}_d(G)$, while the size of a smallest distance-d dominating set in G is denoted by $\mathrm{dom}_d(G)$.

Problem 3. Prove that for every graph G and $d \in \mathbb{N}$, we have

$$\mathrm{ind}_d(G) \geqslant \mathrm{dom}_d(G) \geqslant \mathrm{ind}_{2d}(G).$$

Problem 4. Give a sequence of graphs $G_1, G_2, G_3, \ldots$ such that for every $i \in \mathbb{N}$, $\mathrm{ind}_2(G_i) = 1$ and $\mathrm{dom}_1(G_i) \geqslant i$.

Problem 5. Consider the following algorithm applied on a graph G with a vertex ordering σ. For every vertex u, mark the vertex of $\mathrm{WReach}_d[G, \sigma, u]$ that is the smallest in σ. Letting D be the set of all marked vertices, prove that D is a distance-d dominating set of G that satisfies $|D| \leqslant \mathrm{wcol}_{2d}(G, \sigma) \cdot \mathrm{dom}_d(G)$.

Problem 6. Fix a nowhere dense class $\mathcal{C}$ and $d \in \mathbb{N}$. Prove that given $G \in \mathcal{C}$, $A \subseteq V(G)$, and $k \in \mathbb{N}$, one can compute in polynomial time a subset $B \subseteq A$ such that the size of B is bounded by a function of k, and in G there is a distance-d independent set contained in A if and only if there is one contained in B.

Definition 2. Consider the following algorithm for the Distance-d Dominating Set problem: given G and k, is there a distance-d dominating set in G of size at most k. The algorithm iteratively constructs a sequence of *candidates* $D_1, D_2, D_3, \ldots$ — vertex subsets of size at most k — and a sequence of *witnesses* $w_1, w_2, w_3, \ldots$ — single vertices. Having constructed $D_1, \ldots, D_i$ and $w_1, \ldots, w_i$, the algorithm computes D_{i+1} and w_{i+1} as follows.

Candidate Step: Check whether there exists a set of size at most k that distance-d dominates $\{w_1, \ldots, w_i\}$. If no, then terminate the algorithm and conclude that G does not have a distance-d dominating set of size at most k. Otherwise, pick D_{i+1} to be any such set.

Witness Step: Check whether D_{i+1} is a distance-d dominating set of G. If yes, then terminate the algorithm returning D_{i+1}. Otherwise, pick w_{i+1} to be any vertex not dominated by D_{i+1} and proceed to the next round.

Problem 7. Argue that if d is fixed and G is drawn from a fixed nowhere dense class $\mathcal{C}$, then the ith Candidate Step can be implemented in time $i^{\mathcal{O}(k)} \cdot (n+m)$, while every Witness Step can be implemented in time $\mathcal{O}(k(n+m))$.

Problem 8. Show that if d is fixed and G is drawn from a fixed nowhere dense class $\mathcal{C}$, then for $k = 1$ the algorithm terminates after a constant number of rounds.

Problem 9. Show that if d is fixed and G is drawn from a fixed nowhere dense class $\mathcal{C}$, then the algorithm terminates after $k^{\mathcal{O}(1)}$ rounds, and therefore can be implemented so that it runs in time $2^{\mathcal{O}(k \log k)} \cdot (n+m)$.

Problem 10. For a graph G and $d \in \mathbb{N}$, the *dth power* of G, denoted G^d, is the graph on the same vertex set as G where two vertices u, v are considered adjacent if and only if $\mathrm{dist}_G(u, v) \leqslant d$.

Prove that for every nowhere dense class $\mathcal{C}$ and $d \in \mathbb{N}$, there exists an integer k such that for every n-vertex graph $G \in \mathcal{C}$, the graph G^d has at most n^k different maximal cliques.

Problem 11. For $p \in \mathbb{N}$, a family of sets $\mathcal{F}$ is said to have the *p-Helly property* if the following condition holds. For every subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that every p sets in $\mathcal{G}$ have a nonempty intersection, actually the whole subfamily $\mathcal{G}$ has a nonempty intersection as well.

Prove that for every nowhere dense class of graphs $\mathcal{C}$ and $d \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that for every graph $G \in \mathcal{C}$, the family of distance-d balls $\mathrm{Balls}_d(G) = \{N_d^G[u] \colon u \in V(G)\}$ has the p-Helly property.