ALGOMANET Sparsity

Tutorial 3: Low treedepth colorings

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Problem 1. Prove that if $H \preceq G$, then $\mathrm{td}(H) \leqslant \mathrm{td}(G)$.

Problem 2. Prove that a graph of treedepth at most d is $(d-1)$-degenerate.

Problem 3. Give an algorithm that computes the treedepth of an n-vertex graph in time $2^n \cdot \mathrm{poly}(n)$.

Problem 4. Prove that for any graph G, $\mathrm{td}(G) = \mathrm{wcol}_\infty(G)$.

Problem 5. Prove that a tree on n vertices has treedepth at most $\log_2 n$. Infer that an n-vertex graph of treewidth t has treedepth at most $(t+1) \cdot \log n$.

Problem 6. Prove that a planar graph on n vertices has treedepth $\mathcal{O}(\sqrt{n})$.

You may use the following balanced separator statement for planar graphs: in every n-vertex planar graph G there exists a subset of vertices X of size $\mathcal{O}(\sqrt{n})$ such that every connected component of $G - X$ has at most $n/2$ vertices.

Problem 7. Prove that if $\mathcal{C}$ is a graph class such that for every $p \in \mathbb{N}$ there exists $M(p)$ such that every graph $G \in \mathcal{C}$ admits a treedepth-p coloring with $M(p)$ colors, then $\mathcal{C}$ has bounded expansion.

Problem 8. Prove that if a graph admits a treedepth-p coloring with M colors, then it also admits a p-centered coloring with $M \cdot p^{\binom{M}{<p}}$ colors.

Problem 9. Prove that given an n-vertex graph G together with its elimination forest of depth at most d, one can verify whether G is 3-colorable in time $3^d \cdot \mathrm{poly}(n)$ and space $\mathrm{poly}(n)$.

Problem 10. Prove that given a p-vertex graph H and an n-vertex graph G together with its elimination forest of depth at most d, one can verify whether H is a subgraph of G in time $d^{\mathcal{O}(p)} \cdot n$.

Definition 1. For a graph G and $d \in \mathbb{N}$, we define a graph $G^{=d}$ as follows: the vertex set of $G^{=d}$ is the same as that of G, while two vertices u, v are adjacent in $G^{=d}$ if and only if they are at distance *exactly* d in G.

Problem 11. Prove that every graph G of treedepth at most d admits a coloring using at most $2^d - 1$ colors with the following property: for any pair of vertices u and v, if the distance between u and v in G is finite and odd, then u and v receive different colors.

Problem 12. Prove that if $\mathcal{C}$ is a class of bounded expansion and $d \in \mathbb{N}$ is odd, then there is a number M such that for every graph $G \in \mathcal{C}$, the graph $G^{=d}$ admits a proper coloring with at most M colors.