ALGOMANET Sparsity

Tutorial 2: Generalized coloring numbers

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Problem 1. For every $d \in \mathbb{N}$, compute $\mathrm{adm}_d(\text{Forests})$, $\mathrm{scol}_d(\text{Forests})$, and $\mathrm{wcol}_d(\text{Forests})$.

Problem 2. For a class of graph G and $c \in \mathbb{N}$, we define graph $G \bullet K_c$ by taking G and replacing every vertex u with a clique $K(u)$ of size c so that whenever uv is an edge in G, then every vertex of $K(u)$ is adjacent to every vertex of $K(v)$. Prove that if a class $\mathcal{C}$ has bounded expansion and $c \in \mathbb{N}$ is fixed, then the class $\mathcal{C} \bullet K_c \coloneqq \{G \bullet K_c \colon G \in \mathcal{C}\}$ has bounded expansion as well.

Definition 1. For $d \in \mathbb{N}$, a graph G, vertex subset $S \subseteq V(G)$, and vertex $v \in S$, define $b_d(v, S)$ to be the largest cardinality of a family $\mathcal{P}$ of paths in G with the following properties:

- each path $P \in \mathcal{P}$ has length at most d, leads from v to some other vertex of S, and all its internal vertices do not belong to S; and
- for all distinct $P, P' \in \mathcal{P}$, we have $V(P) \cap V(P') = \{v\}$.

Problem 3. Fix $d \in \mathbb{N}$ and let G be a graph. Consider the following procedure of computing a vertex ordering $\sigma = (v_1, \ldots, v_n)$ of G. Assuming $v_{i+1}, \ldots, v_n$ are already defined, let $S_i \coloneqq V(G) - \{v_{i+1}, \ldots, v_n\}$, and choose v_i to be any vertex $v \in S_i$ that minimizes $b_d(v, S_i)$. Prove that if σ is any vertex ordering obtained using this procedure, then σ has the optimum d-admissibility, i.e., $\mathrm{adm}_d(G, \sigma) = \mathrm{adm}_d(G)$.

Problem 4. Give an algorithm with running time $\mathcal{O}(nm)$ that given $r \in \mathbb{N}$, a graph G, vertex subset $S \subseteq V(G)$, and $v \in S$, computes a family of paths $\mathcal{P}_v$ as in the definition of $b_d(v, S)$, but of size at least $\frac{b_d(v,S)}{d}$.

Problem 5. Give an algorithm with runtime $\mathcal{O}(n^3 m)$ that given $d \in \mathbb{N}$ and a graph G, computes a vertex ordering σ of G with $\mathrm{adm}_d(G, \sigma) \leqslant d \cdot \mathrm{adm}_d(G)$.

Problem 6. Let G be a graph and let $d \in \mathbb{N}$. Prove that if $u, v \in V(G)$ are two distinct vertices, then every path of length at most d between u and v is hit by the set

$$\mathrm{WReach}_d[G, \sigma, u] \cap \mathrm{WReach}_d[G, \sigma, v].$$

Problem 7. Let $\mathcal{C}$ be a class of bounded expansion and $d \in \mathbb{N}$ be fixed. Prove that there exists $c \in \mathbb{N}$ such that for every graph $G \in \mathcal{C}$ and $A \subseteq V(G)$, there exists $B \supseteq A$ satisfying the following conditions:

- $|B| \leqslant c \cdot |A|$; and
- for every pair of vertices $u, v \in A$ such that $\mathrm{dist}_G(u, v) \leqslant d$, we have $\mathrm{dist}_{G[D]}(u, v) = \mathrm{dist}_G(u, v)$.

Problem 8. Let $\mathcal{C}$ be a class of bounded expansion and $p \in \mathbb{N}$ be fixed. Prove that there is an integer $c \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ there exists a directed graph D on the same vertex set as G that satisfies the following conditions:

- Every vertex u of D has at most c *outneighbors* in D, that is, vertices v such that the arc (u, v) is present in D.
- For every vertex subset $A \subseteq V(G)$ such that $|A| \leqslant p$ and $G[A]$ is connected, there exists $w \in A$ such that for every $u \in A \setminus \{w\}$ there is an edge (u, w) in D.

Problem 9. Let G be a graph, let $r \in \mathbb{N}$, and let σ be a vertex ordering of G. For every vertex $u \in V(G)$, define the *cluster* of u as follows:

$$C_u \coloneqq \{v \in V(G) : \ u \in \mathrm{WReach}_{2r}[G, \sigma, v]\}.$$

Prove that the following conditions hold:

- each cluster has radius at most $2r$;
- each vertex of $V(G)$ appears in at most $\mathrm{wcol}_{2r}(G, \sigma)$ clusters; and
- for each vertex $u \in V(G)$, the distance-r neighborhood $N_r^G[u]$ of u in G is entirely contained in some cluster.

Such family of clusters is called an *distance-r neighborhood cover* of G with radius $2r$ and overlap $\mathrm{wcol}_{2r}(G, \sigma)$.