Sparsity — tutorial 1

Measuring sparsity

Problem 4. Let G be an n-vertex graph with $K_t \not\preccurlyeq G$. Prove that G has at most $2^t \cdot n$ edges.

The solution of the above exercise has been added to the lecture notes.

Problem 10. Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant c_r such that the following holds. For every graph $G \in \mathcal{C}$ and every subset A of its vertices, there exists a vertex subset $B \supseteq A$ such that $|B| \leqslant c_r|A|$ and for every vertex $u \in V(G) - B$, at most c_r vertices of B can be reached from u by a path of length at most r whose internal vertices do not belong to B.

Solution. We first introduce some convenient notation. For any graph H, any set $X \subseteq V(H)$ and any $u \in V(H) - X$, the set of vertices of X which can be reached from u by a path of length at most r whose internal vertices do not belong to X will be called the *r-projection of u onto X in H*, and denoted by $M_r^H(u, X)$. Thus, we need to prove that there is some superset $B \supseteq A$ whose size is bounded linearly in $|A|$, and such that r-projections onto B have bounded sizes.

Let us fix the constant $\xi = \lceil 2\nabla_{r-1}(\mathcal{C})\rceil$. We consider the following iterative procedure.

1. Start with $H = G$ and $Y = A$. We will maintain the invariant that $Y \subseteq V(H)$.
2. As long as there exists a vertex $u \in V(H) - Y$ with $|M_r^H(u, Y)| \geqslant \xi$ do the following:
   - Select an arbitrary subset $Z_u \subseteq M_r^H(u, Y)$ of size exactly ξ.
   - For each $w \in Z_u$, select a path P_w that starts at u, ends at w, has length at most r, and all its internal vertices are in $V(H) - Y$.
   - Modify H by contracting $\bigcup_{w \in Z_u}(V(P_w) - \{w\})$ onto u, and add the obtained vertex to Y.

Observe that in a round of the procedure above we always make a contraction of a connected subgraph of $H - Y$ of radius at most $r - 1$. Also, the resulting vertex falls into Y and hence does not participate in future contractions. Thus, at each point H is an $(r-1)$-shallow minor of G. For any moment of the procedure and any $u \in V(H)$, by $\tau(u)$ we denote the subset of original vertices of G that were contracted onto u during earlier rounds. Note that either $\tau(u) = \{u\}$ when u is an original vertex of G, or $\tau(u)$ is a set of cardinality at most $1 + (r-1)\xi$.

We claim that the presented procedure stops after at most $|A|$ rounds. Suppose otherwise, that we successfully constructed the graph H and subset Y after $|A|+1$ rounds. Examine graph $H[Y]$. This graph has $2|A|+1$ vertices: $|A|$ original vertices of A and $|A|+1$ vertices that were added during the procedure. Whenever a vertex u is added to Y after contraction, then it introduces at least ξ new edges to $H[Y]$: these are edges that connect the contracted vertex with the vertices of Z_u. Hence, $H[Y]$ has at least $\xi(|A|+1)$ edges, which means that

$$\frac{|E(H[Y])|}{|V(H[Y])|} \geqslant \frac{\xi(|A|+1)}{2|A|+1} > \nabla_{r-1}(\mathcal{C}).$$

This is a contradiction with the fact that H is an $(r-1)$-shallow minor of G.

Therefore, the procedure stops after at most $|A|$ rounds producing (H, Y), where $|M_r^H(u, Y)| < \xi$ for each $u \in V(H) - Y$. Define $B = \tau(Y) = \bigcup_{u \in Y} \tau(u)$. Obviously, we have $A \subseteq B$. Since $|\tau(u)| = 1$ for each original vertex $u \in A$ and $|\tau(u)| \leqslant 1 + (r-1)\xi$ for each u that was added during the procedure, we have $|B| \leqslant ((r-1)\xi + 2) \cdot |A|$. We are left with proving that r-projections are small.

By construction, we have $V(H) - Y = V(G) - B$. Take any $u \in V(H) - Y$ and observe that $M_r^G(u, B) \subseteq \tau(M_r^H(u, Y))$. Since $|M_r^H(u, Y)| < \xi$ for each $u \in V(H) - Y$ and $|\tau(u)| \leqslant 1 + (r-1)\xi$ for each $u \in V(H)$, we have $|M_r^G(u, B)| \leqslant \xi(1 + (r-1)\xi)$. Hence, we may conclude by defining $c_r = \xi((r-1)\xi + 2)$. □