Sparsity — tutorial 1

Measuring sparsity

Problem 1. Prove that a graph is 1-degenerate if and only if it is a forest.

Problem 2. Prove that every d-degenerate graph is $(d+1)$-colorable.

Problem 3. Prove that every planar graph is 5-degenerate.

Problem 4. Let G be an n-vertex graph with $K_t \not\preccurlyeq G$. Prove that G has at most $2^t \cdot n$ edges.

Problem 5. Prove that for a graph class $\mathcal{C}$, the following conditions are equivalent:

- There is a constant $c \in \mathbb{N}$ such that $\nabla_r(\mathcal{C}) \leqslant c$ for all $r \in \mathbb{N}$.
- There is a graph H such that H is not a minor of any graph from $\mathcal{C}$.

Problem 6. Prove that every graph on 2^{a+b} vertices contains either a clique of size a or an independent set of size b. Conclude that there is a function $N(p,k)$ such that every complete graph on $N(p,k)$ vertices with edges colored with p colors contains a monochromatic clique on k vertices.

Problem 7. Prove that the following statements for a graph class $\mathcal{C}$ are equivalent:

- The class $\mathcal{C}$ is somewhere dense.
- There is $r \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, the r-subdivision of K_n is a subgraph of some graph from $\mathcal{C}$.

Problem 8. Prove that if G is an n-vertex graph of degeneracy d, then for every $c \in \mathbb{N}$ the graph G contains at most n/c vertices of degree at least $2cd$.

Problem 9. Prove that if G is a graph of degeneracy d and A is a subset of its vertices, then there exists a vertex subset $B \supseteq A$ such that $|B| \leqslant 2|A|$ and every vertex of $V(G) - B$ has at most $2d$ neighbors in B.

Problem 10. Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant c_r such that the following holds. For every graph $G \in \mathcal{C}$ and every subset A of its vertices, there exists a vertex subset $B \supseteq A$ such that $|B| \leqslant c_r|A|$ and for every vertex $u \in V(G) - B$, at most c_r vertices of B can be reached from u by a path of length at most r whose internal vertices do not belong to B.