Sparsity — open problems

Problem 1 (Fine bounds for strong coloring number)**.** Does there exist a polynomial $P(\cdot,\cdot)$ such that for every graph G and $r \in \mathbb{N}$ we have

$$\mathrm{col}_r(G) \leqslant P(r, \nabla_r(G)).$$

Note that the degree of P has to be independent of r.

Status: The statement holds for r-admissibility and does not hold for the weak r-coloring number.

Problem 2 (Fine bounds for admissibility)**.** Does there exist a sublinear function $f(\cdot)$ such that for every planar graph G and $r \in \mathbb{N}$ we have

$$\mathrm{adm}_r(G) \leqslant f(r).$$

Status: The currently best bound for the r-admissibility on planar graphs is $c \cdot r$ for some small constant c. Some indication that it could be sublinear is given in [Weißauer, *On the block number of graphs*].

Problem 3 (Easy linear-time approximation of generalized coloring numbers)**.** Find a simple algorithm that given a graph G with weak $g(r)$-coloring number bounded by d, computes a vertex ordering of G with weak r-coloring bounded by $h(d)$, for some functions g, h. The algorithm should run in time $f(r,d) \cdot n$ for some function f, where $n = |V(G)|$.

Status: There is a linear-time algorithm for this problem that computes r-admissibility exactly, but to achieve linear running time one needs to rely on a complicated data structure. Greedily approximating r-admissibility as shown during the tutorials, without any tricks, incurs cubic running time. Another approach is based on so-called transitive fraternal augmentations, however, this approach turns out to be completely impractical for larger values of r. The goal is to obtain something simple and possibly implementable.

Problem 4 (Approximation for treedepth)**.** Determine whether there exists an algorithm that given a graph G and integer d, works in time $2^{\mathcal{O}(d)} \cdot n^{\mathcal{O}(1)}$ and either concludes that the treedepth of G is larger than d, or finds a treedepth decomposition of G of depth at most $c \cdot d$, for any universal constant c.

Status: Current approaches yield either an exact FPT algorithm with running time $2^{\mathcal{O}(d^2)} \cdot n$, or a polynomial-time algorithm yielding a decomposition of depth $\mathrm{poly}(d)$. The challenge is to keep both a single-exponential FPT running time and get a constant-factor approximation, but any weaker result (say, an $\mathcal{O}(\log \mathsf{OPT})$-approximation) would be also interesting.

Problem 5 (Bounding the number of rounds of iterative approach to r-DomSet)**.** In Problem 3 of Tutorial 9 we gave a simple iterative FPT algorithm for r-Dominating Set on any nowhere dense class $\mathcal{C}$ and integer $r \in \mathbb{N}$. We proved that the number of rounds is bounded by a polynomial of the target size k, where the degree of the polynomial depends (quite badly) on $\mathcal{C}$ and r. Give more precise estimates on the number of rounds on specific classes of sparse graphs, say forests, planar graphs, and, more generally, H-minor-free for a fixed H. In particular, does the degree of the polynomial need to depend on r?

Problem 6 was removed, because it turned out that a solution essentially follows from known heavy tools.

Problem 7 (Constant factor approximation for weighted distance-r dominating set on classes of bounded expansion.)**.** Let $\mathcal{C}$ be a graph class of bounded expansion and let $r \in \mathbb{N}$ be fixed. Assume every graph G is equipped with a weight function $w\colon V(G) \to \mathbb{N}$. Can we find for every graph $G \in \mathcal{C}$ in polynomial time a constant factor approximation of a minimum weight distance-r dominating set? The unweighted case was discussed in the lectures.

Problem 8 (Bounded expansion for infinite graphs)**.** Similarly to the notion of nowhere denseness for classes of infinite graphs, one can also consider generalized coloring numbers for infinite graphs by measuring admissibility and weak/strong coloring number for any linear order of the vertices. Are the generalized coloring numbers equivalent to density of depth-r minors also for infinite graphs, similarly as in the finite setting? What should be the right definitions, maybe one would like to work on well-orderings of the vertex set? Can one prove statements about classes of finite graphs of bounded expansion by a translation to the infinite setting using Łoś' theorem?

Problem 9 (Robust low treedepth colorings for nowhere dense classes)**.** Suppose G is a graph and λ is a coloring of G with a set of colors C. Define the *coloring extension* G^{λ} as the graph obtained from G as follows: for each color $c \in C$ introduce a fresh vertex u_c and make it adjacent to all vertices of G that have color c in λ.

A class $\mathcal{D}$ of graphs has *of bounded VC-dimension* if there is a function $k\colon \mathbb{N} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$ and $G \in \mathcal{D}$, the family

$$\mathrm{Balls}_r(G) = \{\{u\colon \mathrm{dist}(u,v) \leqslant r\}\colon u \in V(G)\}$$

has VC-dimension bounded by $k(r)$.

Prove or disprove the following statement. Let $\mathcal{C}$ be a nowhere dense class of graphs. Then for every $p \in \mathbb{N}$ there exists a constant k and, for every $\epsilon > 0$, a constant c_ϵ, such that every n-vertex graph $G \in \mathcal{C}$ admits a treedepth-p coloring λ_G using $c_\epsilon \cdot n^\epsilon$ colors such that the class $\{G^{\lambda_G}\colon G \in \mathcal{C}\}$ has bounded VC-dimension.

Problem 10 (Actual edge density in nowhere dense classes)**.** Prove or disprove the following statement: If $\mathcal{C}$ is nowhere dense, then every n-vertex graph from $\mathcal{C}$ has at most $n \cdot \mathrm{poly}(\log n)$ edges.

Status: The upper bound is $\mathcal{O}(n^{1+\epsilon})$ for any $\epsilon > 0$, while known constructions of nowhere dense classes without bounded expansion yield only logarithmic lower bounds on the edge density. If true, this statement would be a breakthrough.

Problem 11 (Quasi-planar graphs)**.** A graph G is called *k-quasi-planar* if there is an embedding of G into the plane where there are no k edges that are pairwise intersect. The status of the following statements is open:

- The class of 3-quasi-planar graphs has bounded expansion.
- The class of k-quasi-planar graphs has bounded degeneracy, for any $k > 4$.

It is currently known that the classes of 3- and 4-quasi-planar graphs are c-degenerate for some constant c, and that an n-vertex k-quasi-planar graph has $\mathcal{O}(n \log n)$ edges, for every fixed k. Also, it is not that hard (though non-trivial) to show that the class of k-quasi-planar graphs is nowhere dense, for every k. Thus, k-quasi-planar graphs are an interesting example of a class whose sparsity is very plausible, but not yet proved.

Problem 12 (Almost Strong Erdős-Hajnal for interpretations of nowhere dense classes)**.** Prove or disprove the following conjecture. Let $\mathcal{C}$ be a nowhere dense class and let $\varphi(x, y)$ be a first order formula with two free variables. Then for every $\epsilon > 0$ there exists N such that for every graph $G \in \mathcal{C}$ with $n > N$ vertices, G contains two disjoint subsets A and B, each on at least $n^{1-\epsilon}$ vertices, such that A and B are *φ-homogeneous* in the following sense: either $\varphi(a, b)$ holds for all $a \in A$ and $b \in B$, or $\varphi(a, b)$ does not hold for all $a \in A$ and $b \in B$.