Sparsity — homework 6

VC dimension, polynomial expansion, deadline: February 1st, 2018, 14:15 CET

Problem 1. A graph class $\mathcal{C}$ has *bounded doubling dimension* if there exists a number $M \in \mathbb{N}$ such that the following holds: for every graph $G \in \mathcal{C}$ and every integer $r \geqslant 1$, every ball of radius r in G can be covered by at most M balls of radius $\lfloor r/2 \rfloor$. Prove that every class with bounded doubling dimension has polynomial expansion.

Note: A ball of radius r in a graph G is a set of vertices of the form $\{u \colon \mathrm{dist}_G(u, v) \leqslant r\}$ for some $v \in V(G)$. Note that balls of radius 0 are single vertices.

Problem 2. Consider the λ-local search algorithm for the r-Independent Set problem: start with an arbitrary solution and as long as there is an improvement step consisting of removing and adding at most λ vertices to the current solution, apply it.

Prove that for every $r \in \mathbb{N}$, $\epsilon > 0$, and class $\mathcal{C}$ of polynomial expansion there exists a constant $\lambda \in \mathbb{N}$, depending only on r, ϵ, and $\mathcal{C}$, such that λ-local search yields a $(1-\epsilon)$-approximation algorithm for r-Independent Set on graphs from $\mathcal{C}$. More precisely, λ-local search applied on any graph $G \in \mathcal{C}$ computes an r-independent set of size at least $1 - \epsilon$ times the largest size of an r-independent set in G.

Problem 3. Fix $d \in \mathbb{N}$. Consider the set system $\mathcal{F}$ of *half-spaces* over the vector space $\mathbb{R}^d$: for each non-zero vector $w \in \mathbb{R}^d$ and $c \in \mathbb{R}$, include in $\mathcal{F}$ the half-space

$$A_{w,c} \coloneqq \{v \in \mathbb{R}^d \colon \langle v, w \rangle \leqslant c\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. Prove that the VC dimension of $\mathcal{F}$ is equal to $d + 1$.

Note: The definition of VC dimension works equally well when the ground set is infinite. Also, note that the task consists of an upper and a lower bound.