Sparsity — homework 5

Uniform quasi-wideness, Splitter game, deadline: December 18th, 2017, 14:15 CET

Problem 1. Prove that for every graph G and $r \in \mathbb{N}$, Splitter wins the $(\mathrm{wcol}_{2r}(G), 1, r)$-Splitter game on G.

Problem 2. Prove that for every nowhere dense class $\mathcal{C}$, integer $r \in \mathbb{N}$, and real $\delta > 0$ there exists an integer $M \in \mathbb{N}$ such that the following holds. For every graph $G \in \mathcal{C}$ and subset of vertices $A \subseteq V(G)$ such that $|A| \geqslant M$ and $\mathrm{dist}_G(u, v) \leqslant 2r$ for all $u, v \in A$, there exists a set $D \subseteq V(G)$ that r-dominates A and has size at most $\delta|A|$.

Problem 3. Suppose $r \in \mathbb{N}$, G is a graph, S is subset of vertices of G, and $(u_1, v_1), (u_2, v_2)$ are two pairs of vertices from G. We say that S *r-separates* (u_1, v_1) and (u_2, v_2) if every path of length at most r with one endpoint in $\{u_1, v_2\}$ and second in $\{u_2, v_2\}$ contains a vertex of S.

Prove that for every nowhere dense class $\mathcal{C}$ and integer $r \in \mathbb{N}$, there exist a constant $s_r \in \mathbb{N}$ and a function $N_r \colon \mathbb{N} \to \mathbb{N}$ such that the following holds. For every $m \in \mathbb{N}$, graph $G \in \mathcal{C}$, and set A of pairs of vertices of G with $|A| \geqslant N_r(m)$, there exist $S \subseteq V(G)$ and $B \subseteq A$ with $|S| \leqslant s_r$ and $|B| \geqslant m$ such that every pair of distinct pairs from B is r-separated by S.