Sparsity — homework 4

Tree-depth, low td-colorings, neighborhood complexity, deadline: December 4th, 2017, 14:15 CET

Problem 1. Prove that for every class of bounded expansion $\mathcal{C}$ and for every $r \in \mathbb{N}$ there exists a real $\delta > 0$ such that the following holds. Suppose $G \in \mathcal{C}$ is a graph and $A \subseteq V(G)$ is a subset of its vertices with the following property: for each pair of distinct vertices $u, v \in A$, there exists a vertex $w \in V(G)$ such that $\mathrm{dist}_G(u, w) \leqslant r$ and $\mathrm{dist}_G(v, w) \leqslant r$. Then there exists a vertex $x \in V(G)$ such that $|N_G^r[x] \cap A| \geqslant \delta\sqrt{|A|}$.

Problem 2. Prove that there exists an algorithm that, given an n-vertex graph G together with its tree-depth decomposition of height at most d, verifies whether G admits a proper 3-coloring in time $\mathcal{O}(3^d \cdot n^c)$ and space $\mathcal{O}(n^c)$, for some constant c independent of d. The constants hidden in the $\mathcal{O}(\cdot)$-notation may **not** depend on d.

Problem 3. For a graph G and $r \in \mathbb{N}$, by $G^{=r}$ we denote the graph on vertex set $V(G)$ where two vertices u and v are adjacent if and only if the distance between them in G is equal *exactly* to r.

Prove that for every odd integer $r \in \mathbb{N}$ and class of bounded expansion $\mathcal{C}$, there exists a number $M \in \mathbb{N}$ such that for every $G \in \mathcal{C}$, the graph $G^{=r}$ admits a proper M-coloring.

Note: A proper k-coloring of a graph is a coloring of its vertices with k colors where no two adjacent vertices receive the same color.