Sparsity — homework 3

Generalized coloring numbers, deadline: November 20th, 2017, 14:15 CET

Problem 1. Let $r \in \mathbb{N}$, let G be a graph, and let σ be a vertex ordering of G. Consider the following algorithm. Every vertex $u \in V(G)$ picks $v(u)$ to be the smallest vertex of $\mathrm{WReach}_r[G, \sigma, u]$ in the ordering σ. Then, define D as the set of those vertices that have been picked by any vertex; that is, $D \coloneqq \{v(u) \colon u \in V(G)\}$.

Prove that D is an r-dominating set of G that moreover satisfies $|D| \leqslant \mathrm{wcol}_{2r}(G, \sigma) \cdot \mathrm{dom}_r(G)$.

Problem 2. Let $\mathcal{I}_k$ be the class of intersection graphs of families of closed intervals on a line with ply at most k. In other words, a graph G belongs to $\mathcal{I}_k$ if and only if we can associate a closed interval $I_u \subseteq \mathbb{R}$ with every vertex $u \in V(G)$ such that $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$, and no $x \in \mathbb{R}$ belongs to more than k intervals from $\{I_u\}_{u \in V(G)}$.

Prove that $\mathrm{wcol}_r(\mathcal{I}_k) \leqslant \binom{r+k-1}{r}$ for all $r \in \mathbb{N}$.

Problem 3. For a graph G, integer $r \in \mathbb{N}$, and a vertex subset $A \subseteq V(G)$, an *r-shortest path closure* of A is any $B \supseteq A$ such that for all $u, v \in A$ with $\mathrm{dist}_G(u, v) \leqslant r$, we have $\mathrm{dist}_{G[B]}(u, v) = \mathrm{dist}_G(u, v)$.

Prove that for every class $\mathcal{C}$ of bounded expansion and integer $r \in \mathbb{N}$, there exists a constant $c \in \mathbb{N}$, depending on $\mathcal{C}$ and r, such that the following holds. For every graph $G \in \mathcal{C}$, one may assign to each vertex $u \in V(G)$ a set $L_u \subseteq V(G)$ with $|L_u| \leqslant c$, such that for every vertex subset $A \subseteq V(G)$, the set $B \coloneqq \bigcup_{u \in A} L_u$ is an r-shortest path closure of A.