Sparsity — homework 2

Measuring sparsity, deadline: November 6th, 2017, 14:15 CET

Problem 1. Prove that a class $\mathcal{C}$ of graphs has bounded expansion if and only if the following condition holds. For every $r \in \mathbb{N}$ there exists a constant $c_r \in \mathbb{N}$ such that whenever the exact r-subdivision of some graph H is a subgraph of some graph from $\mathcal{C}$, then the average degree of H is at most c_r.

Note: The exact r-subdivision of H is obtained from H by subdividing every its edge exactly r times.

Problem 2. Prove that a class $\mathcal{C}$ of graphs is nowhere dense if and only if the following condition holds. For every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a constant $c_{r,\varepsilon} \in \mathbb{N}$ such that every graph $G \in \mathcal{C}\nabla r$ contains at most $c_{r,\varepsilon} \cdot |V(G)|^{1+\varepsilon}$ distinct cliques.

Problem 3. Let $\mathcal{C}$ be a class of bounded expansion. Prove that for every $t \in \mathbb{N}$ there exists a constant $K = K(t) \in \mathbb{N}$ with the following property. For every graph $G \in \mathcal{C}$ and every vertex subset $A \subseteq V(G)$, there exist vertex subsets $S \subseteq V(G)$ and $B \subseteq A - S$ such that

- $|S| \leqslant |A|/t$,
- $|B| \geqslant |A|/K$, and
- vertices of B are pairwise at distance more than 3 in $G - S$.